Graviton Corrections to Vacuum Polarization during Inflation

Katie E. Leonard[†] and R. P. Woodard[‡]

Department of Physics
University of Florida
Gainesville, FL 32611

ABSTRACT

We use dimensional regularization to compute the one loop quantum gravitational contribution to the vacuum polarization on de Sitter background. Adding the appropriate BPHZ counterterms gives a fully renormalized result which can be used to quantum correct Maxwell's equations. We use the Hartree approximation to argue that the electric field strengths of photons experience a secular suppression during inflation.

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[†] e-mail: katie@phys.ufl.edu
[‡] e-mail: woodard@phys.ufl.edu

Dedicated to Stanley Deser on the occasion of his 82nd birthday.

1 Introduction

Inflation produces a vast ensemble of infrared scalars and gravitons. This is thought to be the source of primordial scalar and tensor perturbations [1]. It is natural to wonder what effect these ensembles have on other particles. That sort of question can be answered by computing the scalar or graviton contribution to the appropriate 1PI (one-particle-irreducible) 2-point function and then using that result to quantum-correct the linearized field equation for the particle in question.

The 1PI 2-point function for a scalar is known as its "self-mass-squared", $-iM^2(x;x')$, and the quantum-corrected, linearized field equation for a massless, minimally coupled scalar is,

$$\partial_\mu\Big(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\varphi(x)\Big) - \int d^4x' M^2(x;x')\varphi(x') = 0 \;, \tag{1}$$

where $g_{\mu\nu}(x)$ is the spacelike metric tensor. The fermion's 1PI 2-point function is called its "self-energy", $-i[_i\Sigma_j](x;x')$, and the quantum-corrected, linearized field equation for a massless fermion is,

$$\sqrt{-g}\,e^\mu_a\gamma^a_{ij}\Big(i\partial_\mu\delta_{jk}-\frac{1}{2}A_{\mu bc}J^{bc}_{jk}\Big)\psi_k(x) - \int d^4x'\Big[_i\Sigma_j\Big](x;x')\psi_j(x') = 0 \;, \tag{2}$$

where $e_{\mu a}(x)$ is the vierbein field, γ^a_{ij} are the gamma matrices, $A_{\mu bc}(x)$ is the spin connection, and $J^{bc} \equiv -\frac{i}{4}[\gamma^b,\gamma^c]$ are the Lorentz generators. The 1PI 2-point function for a photon has the evocative name "vacuum polarization", $+i[^\mu\Pi^\nu](x;x')$, and the quantum-corrected Maxwell equation is,

$$\partial_\nu\Big(\sqrt{-g}\,g^{\nu\rho}g^{\mu\sigma}F_{\rho\sigma}(x)\Big) + \int d^4x'\Big[^\mu\Pi^\nu\Big](x;x')A_\nu(x') = J^\nu(x) \;, \tag{3}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor and $J^\mu(x)$ is the current density. And the 1PI 2-point function for a graviton is termed the "graviton self-energy", $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$, and the quantum-corrected, linearized Einstein equation is,

$$\sqrt{-g}\,\mathcal{D}^{\mu\nu\rho\sigma}h_{\rho\sigma}(x) - \int d^4x'\Big[^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x')h_{\rho\sigma}(x') = \frac{1}{2}\kappa\sqrt{-g}\,T^{\mu\nu}_{\text{lin}}(x) \;, \tag{4}$$

where $\mathcal{D}^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator, $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity and $T^{\mu\nu}_{\text{lin}}$ is the linearized stress tensor.

Many results of this type have been derived in recent years, with the background geometry of primordial inflation modeled using the cosmological patch of de Sitter space. The effects of massless, minimally coupled (MMC) scalars are simplest to study. A quartic self-interaction leads MMC scalars to develop a growing mass [2]. The vacuum polarization from charged MMC scalars causes the photon to develop a mass [3] and engenders profound changes in electrodynamic forces [4]. MMC scalars which are Yukawa-coupled to a fermion make the fermion develop a growing mass [5]. And MMC scalars do not have any effect on gravitons provided one can absorb certain surface terms into perturbative corrections of the initial state [6].

The effects of inflationary gravitons are more difficult to work out (everything is tougher in quantum gravity!), but studies have been made of what they do to fermions and to MMC scalars. The results are interestingly different: whereas inflationary gravitons induce a slow growth of the fermion field strength [7] they have no secular effect on MMC scalars [8]. The difference seems to be due to spin. A MMC scalar can only interact with gravitons through its kinetic energy but this cannot mediate any secular growth, in spite of the growing graviton field strength, because the scalar's kinetic

energy redshifts to zero exponentially fast. By contrast, a fermion interacts with gravitons through its spin, in addition to its kinetic energy, and the spin-spin interaction remains effective even when the kinetic energy redshifts to zero [9]. The same thing seems to be true of a small mass [10].

The importance of spin in mediating interactions between inflationary gravitons and massless fermions suggests that there might be comparably strong effects on other particles with spin such as photons and gravitons. The graviton contribution to the one loop graviton self-energy has been worked out [11] but so far not used to quantum-correct the linearized Einstein equation. The purpose of this paper is to derive the one loop graviton contribution to the vacuum polarization on de Sitter background. We will use it to solve the quantum-corrected Maxwell equation in a subsequent paper. (The flat space analog of this problem was carried out as a warmup exercise [12].) Our computation is done in dimensional regularization, fully renormalized with the necessary BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann) counterterms [13], and reported in the noncovariant tensor basis whose efficacy has been demonstrated in a recent study [14]. We also use the Hartree approximation to argue that photons likely experience a secular suppression of their electric field strengths.

This paper contains seven sections of which the first is this Introduction. Section 2 gives those of the Feynman rules of Maxwell + Einstein which are needed for our computation. The contribution from a single 4-point vertex is derived in section 3. Section 4 gives the much more complicated contribution from two 3-point vertices. Renormalization is accomplished in section 5. Although the use of our result to quantum-correct Maxwell's equation is deferred to a latter work, the Hartree approximation is employed in section 6 to argue that photons experience secular changes of the same strength but opposite sign as those of fermions [7, 9]. Our conclusions are given in section 7.

2 Feynman Rules

The purpose of this section is to present the formalism used to compute the one loop quantum gravity contribution to the vacuum polarization depicted in Fig. 1. We begin by describing the background geometry. Then we use the primitive Lagrangians to derive formal expressions for the first two diagrams of Fig. 1. The longest subsection discusses our conventions for gauge fixing and the resulting propagators. We next describe how the vacuum polarization can be represented in terms of two structure functions. The section closes by giving the counterterms needed for this computation.

2.1 Our de Sitter Background

We model primordial inflation as the cosmological patch of de Sitter space. The invariant element is,

$$ds^2 = a^2 \left[-d\eta^2 + d\vec{x} \cdot d\vec{x} \right] , \tag{5}$$

where $a(\eta) = -\frac{1}{H\eta} = e^{Ht}$ is the scale factor and H is the Hubble parameter. Whereas the spatial coordinates $\vec{x}$ take their usual values, the conformal time η runs from $\eta \to -\infty$ (the infinite past) to $\eta \to 0^-$ (the infinite future).

In representing functions such as propagators which depend upon two points, x^μ and x'^μ, we will make extensive use of the de Sitter length function,

$$y(x; x') \equiv a(\eta)a(\eta')H^2 \left[||\vec{x} - \vec{x}'||^2 - (|\eta - \eta'| - i\delta)^2 \right] . \tag{6}$$

We also need the de Sitter breaking product of the scale factors a at x^μ and a' at x'^μ,
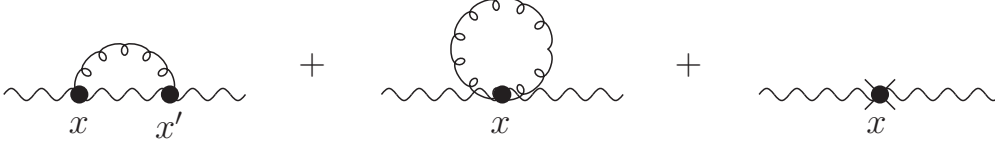
$$u \equiv \ln(aa') . \tag{7}$$

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Figure 1: Graviton contributions to the one loop vacuum polarization. Photon propagators are wavy and graviton propagators are curly.

Derivatives of y and u furnish a convenient basis for representing bi-vector functions of x^μ and x'^μ such as the vacuum polarization,

$$\partial_\mu y \quad , \quad \partial'_\nu y \quad , \quad \partial_\mu \partial'_\nu y \quad , \quad \partial_\mu u \quad , \quad \partial'_\nu u \, . \tag{8}$$

It turns out that either taking covariant derivatives of any of the five basis tensors (8), or contracting any two of them into one another, produces metrics and more basis tensors [15, 16].

2.2 Our Primitive Diagrams

The total Lagrangian consists of the primitive contributions from general relativity and electromagnetism, plus the BPHZ counterterms necessary for this computation,

$$\mathcal{L} = \mathcal{L}_{\rm GR} + \mathcal{L}_{\rm EM} + \mathcal{L}_{\rm BPHZ} \, . \tag{9}$$

The primitive Lagrangians of general relativity and electromagnetism are,

$$\mathcal{L}_{\rm GR} = \frac{1}{16\pi G} \Big(R - (D-2)\Lambda \Big) \sqrt{-g} \quad , \quad \mathcal{L}_{\rm EM} = -\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} \, . \tag{10}$$

The symbol G stands for Newton's constant, while $\Lambda \equiv (D-1)H^2$ is the cosmological constant. We employ a D-dimensional, spacelike metric $g_{\mu\nu}$, with inverse $g^{\mu\nu}$ and determinant $g = \det(g_{\mu\nu})$. Our affine connection and Riemann tensor are,

$$\Gamma^\rho_{\ \mu\nu} \equiv \frac{1}{2} g^{\rho\sigma} \Big[\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} \Big] \, , \tag{11}$$

$$R^\rho_{\ \sigma\mu\nu} \equiv \partial_\mu \Gamma^\rho_{\ \nu\sigma} - \partial_\nu \Gamma^\rho_{\ \mu\sigma} + \Gamma^\rho_{\ \mu\alpha} \Gamma^\alpha_{\ \nu\sigma} - \Gamma^\rho_{\ \nu\alpha} \Gamma^\alpha_{\ \mu\sigma} \, . \tag{12}$$

Our Ricci tensor is $R_{\mu\nu} \equiv R^\rho_{\ \mu\rho\nu}$ and the associated Ricci scalar is $R \equiv g^{\mu\nu} R_{\mu\nu}$. The electromagnetic field strength tensor and its first covariant derivative are,

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \quad , \quad D_\alpha F_{\mu\nu} \equiv \partial_\alpha F_{\mu\nu} - \Gamma^\gamma_{\ \alpha\mu} F_{\gamma\nu} - \Gamma^\gamma_{\ \alpha\nu} F_{\mu\gamma} \, . \tag{13}$$

We define the graviton field $h_{\mu\nu}(x)$ as the difference between the full metric and its de Sitter background value $a^2 \eta_{\mu\nu}$,

$$g_{\mu\nu}(x) \equiv a^2(\eta) \Big[\eta_{\mu\nu} + \kappa h_{\mu\nu}(x) \Big] \equiv a^2 \widetilde{g}_{\mu\nu}(x) \, , \tag{14}$$

where $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity. We follow the usual conventions whereby a comma denotes ordinary differentiation, the trace of the graviton field is $h \equiv \eta^{\mu\nu} h_{\mu\nu}$, and

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graviton indices are raised and lowered using the Minkowski metric, $h^\mu{}_\nu \equiv \eta^{\mu\rho} h_{\rho\nu}$ and $h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}$. Up to a surface term the gravitational Lagrangian can be written as,

$$\mathcal{L}_{\mathrm{GR}} - \mathrm{Surface} = \frac{a^{D-2}}{16\pi G} \sqrt{-\widetilde{g}}\, \widetilde{g}^{\alpha\beta}\widetilde{g}^{\rho\sigma}\widetilde{g}^{\mu\nu}$$

$$\times \left\{ \frac{1}{2} h_{\alpha\rho,\mu} h_{\nu\sigma,\beta} - \frac{1}{2} h_{\alpha\beta,\rho} h_{\sigma\mu,\nu} + \frac{1}{4} h_{\alpha\beta,\rho} h_{\mu\nu,\sigma} - \frac{1}{4} h_{\alpha\rho,\mu} h_{\beta\sigma,\nu} \right\}. \tag{15}$$

From Fig. 1 one can see that we only need (15) for the graviton propagator.

The only interactions we require descend from the second variational derivative of the electromagnetic action,

$$\frac{\delta^2 S_{\mathrm{EM}}}{\delta A_\mu(x)\delta A_\rho(x')} = \partial_\kappa \left\{ \sqrt{-g(x)} \left[g^{\kappa\lambda}(x) g^{\mu\rho}(x) - g^{\kappa\rho}(x) g^{\lambda\mu}(x) \right] \partial_\lambda \delta^D(x-x') \right\}. \tag{16}$$

The necessary vertex functions are obtained by expanding the metric factors,

$$\sqrt{-g} \left(g^{\kappa\lambda} g^{\mu\rho} - g^{\kappa\rho} g^{\lambda\mu} \right) \equiv a^{D-4} \left(\eta^{\kappa\lambda}\eta^{\mu\rho} - \eta^{\kappa\rho}\eta^{\lambda\mu} \right)$$

$$+ \kappa a^{D-4} V^{\mu\rho\kappa\lambda\alpha\beta} h_{\alpha\beta} + \kappa^2 a^{D-4} U^{\mu\rho\kappa\lambda\alpha\beta\gamma\delta} h_{\alpha\beta} h_{\gamma\delta} + O(\kappa^3). \tag{17}$$

The tensor factors for the 3-point and 4-point vertices are,

$$V^{\mu\rho\kappa\lambda\alpha\beta} = \eta^{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu} + 4\eta^{\alpha)[\mu}\eta^{\kappa][\rho}\eta^{\lambda](\beta}, \tag{18}$$

$$U^{\mu\rho\kappa\lambda\alpha\beta\gamma\delta} = \left[\frac{1}{4}\eta^{\alpha\beta}\eta^{\gamma\delta} - \frac{1}{2}\eta^{\alpha(\gamma}\eta^{\delta)\beta} \right] \eta^{\kappa[\lambda}\eta^{\rho]\mu} + \eta^{\alpha\beta}\eta^{\gamma)[\mu}\eta^{\kappa][\rho}\eta^{\lambda](\delta}$$

$$+ \eta^{\gamma\delta}\eta^{\alpha)[\mu}\eta^{\kappa][\rho}\eta^{\lambda](\beta} + \eta^{\kappa(\alpha}\eta^{\beta)[\lambda}\eta^{\rho](\gamma}\eta^{\delta)\mu} + \eta^{\kappa(\gamma}\eta^{\delta)[\lambda}\eta^{\rho](\alpha}\eta^{\beta)\mu} + \eta^{\kappa(\alpha}\eta^{\beta)(\gamma}\eta^{\delta)[\lambda}\eta^{\rho]\mu}$$

$$+ \eta^{\kappa(\gamma}\eta^{\delta)(\alpha}\eta^{\beta)[\lambda}\eta^{\rho]\mu} + \eta^{\kappa[\lambda}\eta^{\rho](\alpha}\eta^{\beta)(\gamma}\eta^{\delta)\mu} + \eta^{\kappa[\lambda}\eta^{\rho](\gamma}\eta^{\delta)(\alpha}\eta^{\beta)\mu}. \tag{19}$$

Parenthesized indices are symmetrized and indices enclosed in square brackets are anti-symmetrized.

If we call the graviton propagator $i[_{\alpha\beta}\Delta_{\gamma\delta}](x;x')$ and the photon propagator $i[_\rho\Delta_\sigma](x;x')$, we can give formal expressions for the first two diagrams of Fig. 1. The one constructed from a single 4-point vertex is,

$$i\left[{}^\mu\Pi^\nu_{\mathrm{4pt}}\right](x;x') = \partial_\kappa \left\{ i\kappa^2 a^{D-4} U^{\mu\nu\kappa\lambda\alpha\beta\gamma\delta}\, i\left[_{\alpha\beta}\Delta_{\gamma\delta}\right](x;x)\, \partial_\lambda \delta^D(x-x') \right\}. \tag{20}$$

The diagram constructed from two 3-point vertices is,

$$i\left[{}^\mu\Pi^\nu_{\mathrm{3pt}}\right](x;x') = \partial_\kappa \partial'_\theta \left\{ i\kappa a^{D-4} V^{\mu\rho\kappa\lambda\alpha\beta}\, i\left[_{\alpha\beta}\Delta_{\gamma\delta}\right](x;x') \right.$$

$$\left. \times i\kappa a'^{D-4} V^{\nu\sigma\theta\phi\gamma\delta}\, \partial_\lambda \partial'_\phi i\left[_\rho\Delta_\sigma\right](x;x') \right\}. \tag{21}$$

2.3 Our Propagators

The quadratic part of the gravitational Lagrangian (15) is,

$$\mathcal{L}^{(2)}_{\mathrm{GR}} = a^{D-2} \left\{ \frac{1}{2} h^{\rho\sigma,\mu} h_{\mu\sigma,\rho} - \frac{1}{2} h^{\mu\nu}{}_{,\mu} h_{,\nu} + \frac{1}{4} h^{,\mu} h_{,\mu} - \frac{1}{4} h^{\rho\sigma,\mu} h_{\rho\sigma,\mu} \right\}. \tag{22}$$

Before fixing the gauge and giving the graviton propagator we must digress to summarize the long and confusing debate between cosmologists and mathematical physicists concerning the de Sitter invariance of free gravitons [17, 18]. Although the propagator equation can be made de Sitter invariant by an appropriate choice of gauge, that does not guarantee the de Sitter invariance of the solution. The classic counter-example — which plays an important role in our solution for the graviton propagator — is the propagator $i\Delta_A(x; x')$ of a massless, minimally coupled scalar,

$$\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu i\Delta_A(x; x')\right) \equiv \sqrt{-g}\,\square i\Delta_A(x; x') = i\delta^D(x-x') \; . \tag{23}$$

Equation (23) is de Sitter invariant, but there is no de Sitter invariant solution for $i\Delta_A(x; x')$ [19]. This can be seen from the time dependence of the coincidence limit [20],

$$i\Delta_A(x; x) = \left(\text{Divergent Constant}\right) + \frac{H^2}{4\pi^2} \times \ln(a) \; . \tag{24}$$

If one chooses the "E(3)" vacuum [21] to preserve the spatial homogeneity and isotropy of cosmology then the unique solution is [22],

$$\begin{aligned}
i\Delta_A(x; x') &= i\Delta_{\text{cf}}(x; x') \\
&+ \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\left\{\frac{D}{D-4}\frac{\Gamma^2(\frac{D}{2})}{\Gamma(D-1)}\left(\frac{4}{y}\right)^{\frac{D}{2}-2} - \pi\cot\left(\frac{\pi}{2}D\right) + \ln(aa')\right\} \\
&+ \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\sum_{n=1}^{\infty}\left\{\frac{1}{n}\frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})}\left(\frac{y}{4}\right)^n - \frac{1}{n-\frac{D}{2}+2}\frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+2)}\left(\frac{y}{4}\right)^{n-\frac{D}{2}+2}\right\},
\end{aligned} \tag{25}$$

where $i\Delta_{\text{cf}}(x; x')$ is the (de Sitter invariant) propagator of a conformally coupled scalar,

$$i\Delta_{\text{cf}}(x; x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(\frac{D}{2}-1\right)\left(\frac{4}{y}\right)^{\frac{D}{2}-1} \; . \tag{26}$$

The massless, minimally coupled scalar is especially relevant to gravitons because Grishchuk showed that the physical components of $h_{\mu\nu}$ (transverse, traceless and purely spatial) obey the same equation [23]. Hence the graviton propagator must break de Sitter invariance as well. A cosmologist would also see this from the scale invariance of the tensor power spectrum [18]. Mathematical physicists for years disputed that conclusion because they found manifestly de Sitter invariant solutions to the propagator equation which results from adding de Sitter invariant gauge fixing functions to the quadratic Lagrangian (22) [24].

The discordant viewpoints have recently converged somewhat with the demonstration that there is an obstacle to adding invariant gauge fixing functions on any manifold, like de Sitter, which possesses a linearization instability [25]. That still leaves open the possibilities of either adding a noninvariant gauge fixing function or else enforcing a de Sitter invariant gauge condition as a strong operator equation. When either possibility is pursued, along with the requirement that the resulting propagator can be expressed as a superposition of plane wave mode functions [26, 27, 28], the result is a de Sitter breaking solution of precisely the form implied by the scale invariance of the tensor power spectrum [29, 30]. However, when a de Sitter invariant gauge condition is employed, in conjunction with analytic continuation techniques (either from Euclidean de Sitter space, or in the mass-squared of certain scalar propagators), the result is de Sitter invariant [31, 32], except for certain discrete choices of the gauge condition.

It is well known that analytic continuation fails to recover power law infrared divergences [33, 34, 16]. In this context the discrete problems, which have long been noted [35, 36], seem suspiciously like the special values at which a power law infrared divergence — which is always present — happens to become logarithmic and hence visible to an analytic continuation technique. We will therefore employ a de Sitter breaking graviton propagator, which seems to be the safer choice. Despite the continuing disagreements, it is important to note that very little difference remains between cosmologists and mathematical physicists. In particular, the various de Sitter breaking solutions for the graviton propagator all give the same result for the linearized Weyl-Weyl correlator [37, 38], and this agrees with the result from de Sitter invariant solutions [39] once some errors are corrected [40].

We fix the gauge by adding [26],

$$\mathcal{L}_{\mathrm{GRfix}} = -\frac{1}{2}a^{D-2}\eta^{\mu\nu}F_\mu F_\nu \ , \ F_\mu \equiv \eta^{\rho\sigma}\Big(h_{\mu\rho,\sigma} - \frac{1}{2}h_{\rho\sigma,\mu} + (D-2)Hah_{\mu\rho}\delta^0_\sigma\Big). \tag{27}$$

The resulting graviton propagator can be expressed as a sum of constant tensor factors multiplied by scalar propagators [26],

$$i\Big[_{\mu\nu}\Delta_{\rho\sigma}\Big](x;x') = \sum_{I=A,B,C}\Big[_{\mu\nu}T^I_{\rho\sigma}\Big] \times i\Delta_I(x;x') \ . \tag{28}$$

We have already seen the de Sitter breaking A-type propagator (25). The B-type and C-type propagators obey the equations,

$$\Big[\Box - (D-2)H^2\Big]i\Delta_B(x;x') = \frac{i\delta^D(x-x')}{\sqrt{-g}} = \Big[\Box - 2(D-3)H^2\Big]i\Delta_C(x;x') \ . \tag{29}$$

Each of these propagators is de Sitter invariant and consists of $i\Delta_{\mathrm{cf}}(x;x')$ plus an infinite series of less singular terms which vanish in $D=4$ dimensions,

$$i\Delta_B(x;x') = B(y) = i\Delta_{\mathrm{cf}}(x;x') - \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\sum_{n=0}^{\infty}\Bigg\{\frac{\Gamma(n+D-2)}{\Gamma(n+\frac{D}{2})}\Big(\frac{y}{4}\Big)^n$$
$$-\frac{\Gamma(n+\frac{D}{2})}{\Gamma(n+2)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2}\Bigg\}, \tag{30}$$

$$i\Delta_C(x;x') = C(y) = i\Delta_{\mathrm{cf}}(x;x') + \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\sum_{n=0}^{\infty}\Bigg\{(n+1)\frac{\Gamma(n+D-3)}{\Gamma(n+\frac{D}{2})}\Big(\frac{y}{4}\Big)^n$$
$$-\Big(n-\frac{D}{2}+3\Big)\frac{\Gamma(n+\frac{D}{2}-1)}{\Gamma(n+2)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2}\Bigg\}. \tag{31}$$

Note that the B-type and C-type propagators agree for $D=4$ dimensions. The tensor factors are,

$$\Big[_{\mu\nu}T^A_{\rho\sigma}\Big] = 2\overline{\eta}_{\mu(\rho}\overline{\eta}_{\sigma)\nu} - \frac{2}{D-3}\overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma} \ , \tag{32}$$

$$\Big[_{\mu\nu}T^B_{\rho\sigma}\Big] = -4\delta^0_{(\mu}\overline{\eta}_{\nu)(\rho}\delta^0_{\sigma)} \ , \tag{33}$$

$$\Big[_{\mu\nu}T^C_{\rho\sigma}\Big] = \frac{2}{(D-2)(D-3)}\Big[(D-3)\delta^0_\mu\delta^0_\nu + \overline{\eta}_{\mu\nu}\Big]\Big[(D-3)\delta^0_\rho\delta^0_\sigma + \overline{\eta}_{\rho\sigma}\Big] \ , \tag{34}$$

where $\overline{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} + \delta^\mu_0\delta^\nu_0$ is the spatial part of the Minkowski metric.

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The quadratic part of the electromagnetic action is,

$$\mathcal{L}_{EM}^{(2)} = -\frac{1}{2}a^{D-4}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}a^{D-4}\partial_\mu A_\nu \partial^\nu A^\mu \; . \tag{35}$$

Of course it is no more possible to add a de Sitter invariant gauge fixing function for electromagnetism than it is for gravity [25]. Unlike gravitons, photons show no physical breaking of de Sitter invariance, so the photon propagator is manifestly de Sitter invariant if one employs an exact de Sitter invariant gauge condition [41]. However, the de Sitter breaking of the graviton propagator implies that there is no point to keeping the photon propagator invariant. We have chosen instead to add the noncovariant gauge fixing function which is most closely related to the gravitational one (27) [42],

$$\mathcal{L}_{GF} = -\frac{1}{2}a^{D-4}\Big(\eta^{\mu\nu}A_{\mu,\nu} - (D{-}4)HaA_0\Big)^2 \; . \tag{36}$$

The associated photon propagator is [42],

$$i\Big[_\mu\Delta_\rho\Big](x;x') = \overline{\eta}_{\mu\rho} \times aa'i\Delta_B(y) - \delta^0_\mu\delta^0_\rho \times aa'i\Delta_C(x;x') \; . \tag{37}$$

2.4   Our Structure Functions

The vacuum polarization is a bi-vector density $i[^\mu\Pi^\nu](x;x')$ which is transverse at each point,

$$\frac{\partial}{\partial x^\mu}\Big[^\mu\Pi^\nu\Big](x;x') = 0 = \frac{\partial}{\partial x'^\nu}\Big[^\mu\Pi^\nu\Big](x;x') \; . \tag{38}$$

Although it possesses 16 components in $D = 4$ dimensions, the combination of transversality (38), reflection invariance — $[^\mu\Pi^\nu](x;x') = [^\nu\Pi^\mu](x';x)$ — and the coordinate symmetries of the vacuum relate these components so that they can be expressed in terms of a very few structure functions. For example, Poincaré invariance implies the following simple form in the flat space limit,

$$\Big[^\mu\Pi^\nu\Big]_{\text{flat}} = \Big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\Big)\partial_\rho\partial'_\sigma\Pi(x{-}x') \; . \tag{39}$$

The one loop fermion and scalar contributions to $\Pi(x - x')$ have been known for decades, and an explicit result for the one loop graviton contribution has recently been derived [12].

Because the graviton vacuum on de Sitter background does not respect full de Sitter invariance, but only spatial homogeneity and isotropy, it turns out that two structure functions are needed [14]. We could still choose to represent the transverse projection operators using covariant derivatives. However, a detailed examination of this form for the already derived vacuum polarization from SQED [3] reveals that it is cumbersome and that it obscures rather than simplifies the essential physics [14]. This seems to be because the conformal invariance of classical electromagnetism in $D = 4$ dimensions is a more powerful organizing principle that the background's isometries. We have therefore chosen to employ the noncovariant form originally used to represent the SQED result,

$$i\Big[^\mu\Pi^\nu\Big](x;x') = \Big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\Big)\partial_\rho\partial'_\sigma F(x;x') + \Big(\overline{\eta}^{\mu\nu}\overline{\eta}^{\rho\sigma} - \overline{\eta}^{\mu\sigma}\overline{\eta}^{\nu\rho}\Big)\partial_\rho\partial'_\sigma G(x;x') \; . \tag{40}$$

We remind the reader that $\overline{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} + \delta^\mu_0\delta^\nu_0$ is the purely spatial part of the Minkowski metric.

By comparing (40) with (39) one sees that our structure function $F(x;x')$ must agree with the flat space result $i\Pi(x - x')$ in the limit that H vanishes with the co-moving time $t = \ln(a)/H$ held

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fixed. Hence the leading divergences are contained in $F(x; x')$. All terms in $G(x; x')$ must contain at least one factor of H^2, and they are correspondingly less divergent. Although our representation (40) is not de Sitter covariant, the two structure functions have very simple physical interpretations in terms of changes in the electric permittivity and the magnetic permeability [43]. However, there is a straightforward procedure for converting our results for $F(x; x')$ and $G(x; x')$ to the physically opaque, de Sitter covariant representation [44] if that is desired.

2.5 Our Counterterms

Deser and van Nieuwenhuizen showed that Einstein + Maxwell is not renormalizable at one loop order [45]. However, it is straightforward to absorb the divergences, order by order, with BPHZ counterterms [13]. We can then solve the quantum-corrected Maxwell equation (3) in the standard sense of effective field theory [46]. This has already been done for quantum gravitational corrections to electrodynamics on flat background [47, 12].

The vacuum polarization has two external photon lines so our computation requires counterterms with two vector potentials. The superficial degree of divergence is four at one loop order, which means there must be four derivatives acting either upon vector potentials or metrics. However, $U(1)$ gauge invariance implies that at least two of the derivatives must act on vector potentials. If both of the remaining two derivatives act upon vector potentials we have the single counterterm which survives in flat space [12],

$$C_4 D_\alpha F_{\mu\nu} D_\beta F_{\rho\sigma} g^{\alpha\beta} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} \ . \tag{41}$$

Because our gauge fixing functions (27) and (36) reduce, for $H \to 0$ at fixed co-moving time, to those employed in our previous flat space computation [12], the divergent part of C_4 must agree as well,

$$C_4 = \frac{\kappa^2}{128\pi^{\frac{D}{2}}} \frac{D\Gamma(\frac{D}{2}-1)}{(D-1)(D-4)} \ . \tag{42}$$

There are three invariant counterterms with two derivatives acting on two vector potentials and two acting on metrics,

$$C_1 F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} R \sqrt{-g} + C_2 F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} R^{\nu\sigma} \sqrt{-g} + C_3 F_{\mu\nu} F_{\rho\sigma} R^{\mu\nu\rho\sigma} \sqrt{-g} \ . \tag{43}$$

However, all the curvatures are related in de Sitter background,

$$R^{\mu\nu\rho\sigma} \longrightarrow H^2 a^{-4} \Big[\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho} \Big] \ , \tag{44}$$

$$R^{\mu\nu} \longrightarrow (D-1) H^2 a^{-2} \eta^{\mu\nu} \ , \tag{45}$$

$$R \longrightarrow (D-1) D H^2 \ . \tag{46}$$

Our computation therefore determines only the combination $\overline{C} \equiv (D-1)D \times C_1 + (D-1) \times C_2 + 2 \times C_3$, and we can write the resulting counterterm as just $\overline{C} H^2 F_{\mu\nu} G_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g}$.

Because our gauge fixing functions (27) and (36) break de Sitter invariance, it is necessary to consider counterterms which are $U(1)$ invariant but not generally coordinate invariant. Two properties of our gauge conditions restrict the number of possibilities:

- They become Poincaré invariant in the flat space limit of $H \to 0$ at fixed co-moving time; and

- They preserve spatial homogeneity and isotropy, as well as the dilatation symmetry $x^\mu \to k x^\mu$.

The first property means that (41) is the only counterterm without explicit factors of H. The second property implies that the only extra counterterm we require has the form, $\Delta C H^2 F_{ij} F_{k\ell} g^{ik} g^{j\ell} \sqrt{-g}$. Our computation therefore requires only three counterterms,

$$\mathcal{L}_{\text{BPHZ}} = \Delta C H^2 F_{ij} F_{k\ell} g^{ik} g^{j\ell} \sqrt{-g} + \overline{C} H^2 F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g}$$
$$+ C_4 D_\alpha F_{\mu\nu} D_\beta F_{\rho\sigma} g^{\alpha\beta} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} . \tag{47}$$

It remains to work out how the counterterms (47) affect the structure functions $F(x; x')$ and $G(x; x')$ in our representation (40) of the vacuum polarization. The first step is taking the variation with respect to $A_\mu(x)$ and $A_\nu(x')$,

$$\frac{\delta S_{\text{BPHZ}}}{\delta A_\mu(x) \delta A_\nu(x')} = -4\Delta C H^2 \partial_\rho \left\{ \sqrt{-g(x)}\, \overline{g}^{\mu\gamma}(x) \overline{g}^{\rho\sigma}(x) \frac{\delta F_{\sigma\gamma}(x)}{\delta A_\nu(x')} \right\}$$
$$+ \partial_\rho \left\{ \sqrt{-g(x)}\, g^{\mu\gamma}(x) g^{\rho\sigma}(x) \left[4\overline{C} H^2 - 4C_4 g^{\alpha\beta}(x) D_\alpha D_\beta \right] \frac{\delta F_{\sigma\gamma}(x)}{\delta A_\nu(x')} \right\} . \tag{48}$$

The next step is specializing to de Sitter, with $g_{\mu\nu} = a^2 \eta_{\mu\nu}$ and $\Gamma^\rho_{\ \mu\nu} = aH(\delta^\rho_\mu \delta^0_\nu + \delta^\rho_\nu \delta^0_\mu - \eta^{\rho 0}\eta_{\mu\nu})$. This is very simple for the ΔC and $\overline{C}$ counterterms,

$$4\Delta C H^2 \partial_\rho \left\{ \sqrt{-g}\, \overline{g}^{\mu\gamma} \overline{g}^{\rho\sigma} \frac{\delta F_{\sigma\gamma}(x)}{\delta A_\nu(x')} \right\} = \left(\overline{\eta}^{\mu\nu} \overline{\eta}^{\rho\sigma} - \overline{\eta}^{\mu\sigma} \overline{\eta}^{\rho\nu} \right) \partial_\rho \partial'_\sigma \left\{ 4\Delta C H^2 a^{D-4} \delta^D(x-x') \right\} , \tag{49}$$

$$4\overline{C} H^2 \partial_\rho \left\{ \sqrt{-g}\, g^{\mu\gamma} g^{\rho\sigma} \frac{\delta F_{\sigma\gamma}(x)}{\delta A_\nu(x')} \right\} = \left(\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\sigma} \eta^{\rho\nu} \right) \partial_\rho \partial'_\sigma \left\{ a^{D-4} \left[4\overline{C} H^2 - 4C_4 \Box \right] \delta^D(x-x') \right\} . \tag{50}$$

Of course (50) contributes directly to the structure function $F(x; x')$, and (49) contributes to $G(x; x')$. The C_4 counterterm is complicated because of the way the tensor d'Alembertian acts on $F_{\sigma\gamma}$,

$$\Box F_{\sigma\gamma} = \frac{1}{a^2} \left\{ \left[\partial^2 - (D-4)Ha\partial_0 + 2(D-2)H^2 a^2 \right] F_{\sigma\gamma} \right.$$
$$\left. -2Ha \left[\delta^0_\sigma \partial^\alpha F_{\alpha\gamma} - \delta^0_\gamma \partial^\alpha F_{\alpha\sigma} \right] + (D-4)H^2 a^2 \left[\delta^0_\sigma F_{0\gamma} - \delta^0_\gamma F_{0\sigma} \right] \right\} . \tag{51}$$

The first term of (51) contributes only to the structure function $F(x; x')$, whereas the second and third terms contribute to both structure functions. After some tedious tensor algebra and reflection of derivatives on the delta function ($\partial_\alpha \delta^D(x - x') = -\partial'_\alpha \delta^D(x - x')$), we find that the counterterms make the following contributions to the two structure functions,

$$\Delta F(x; x') = 4 \left[\overline{C} - (3D-8)C_4 \right] H^2 a^{D-4} i\delta^D(x-x') - 4C_4 a^{D-6} \left[\partial^2 - (D-6)Ha\partial_0 \right] i\delta^D(x-x') , \tag{52}$$

$$\Delta G(x; x') = 4 \left[\Delta C - (D-6)C_4 \right] H^2 a^{D-4} i\delta^D(x-x') . \tag{53}$$

The simplicity of this form is one more indication, added to the many already found [14], of the superiority of the noncovariant representation over the covariant one.

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I	$U_I^{\mu\rho\kappa\lambda\alpha\beta\gamma\delta}$
1	$\frac{1}{4}\eta^{\kappa[\lambda}\eta^{\rho]\mu}\eta^{\alpha\beta}\eta^{\gamma\delta}$
2	$-\frac{1}{2}\eta^{\kappa[\lambda}\eta^{\rho]\mu}\eta^{\alpha(\gamma}\eta^{\delta)\beta}$
3	$\eta^{\alpha\beta}\eta^{\gamma)[\mu}\eta^{\kappa][\rho}\eta^{\lambda](\delta} + \eta^{\gamma\delta}\eta^{\alpha)[\mu}\eta^{\kappa][\rho}\eta^{\lambda](\beta}$
4	$\eta^{\kappa(\alpha}\eta^{\beta)[\lambda}\eta^{\rho](\gamma}\eta^{\delta)\mu} + \eta^{\kappa(\gamma}\eta^{\delta)[\lambda}\eta^{\rho](\alpha}\eta^{\beta)\mu}$
5	$\eta^{\kappa(\alpha}\eta^{\beta)(\gamma}\eta^{\delta)[\lambda}\eta^{\rho]\mu} + \eta^{\kappa(\gamma}\eta^{\delta)(\alpha}\eta^{\beta)[\lambda}\eta^{\rho]\mu}$
6	$\eta^{\kappa[\lambda}\eta^{\rho](\alpha}\eta^{\beta)(\gamma}\eta^{\delta)\mu} + \eta^{\kappa[\lambda}\eta^{\rho](\gamma}\eta^{\delta)(\alpha}\eta^{\beta)\mu}$

Table 1: Parts of 4-point vertex function

3   The 4-Point Contribution

This section summarizes the derivation of the contribution to the vacuum polarization from the 4-point diagram in Fig. 1, by deriving the structure functions required for the desired representation of the vacuum polarization (40). This is achieved by first completing all naive index contractions of (20), followed by a substitution of the graviton propagator, which allows all final index contractions to be taken. Lastly, several contraction identities are introduced from which the 4-point structure functions can be deduced. The procedure outlined in this section will be similar to that used for deriving the 3-point contribution and serves as a simple guide to extracting the desired structure functions.

3.1   Naive Contractions

In order to derive the two scalar structure functions from the 4-point contribution to the vacuum polarization (20) the first step will be to substitute (19) for the 4-point vertex function and apply the naive contractions. This will naturally engender many terms, and to make the process simple for the reader to follow we will break (19) into pieces and then define the 4-point contribution as a sum of those pieces

$$i\left[{}^{\mu}\Pi_{4\text{pt}}^{\nu}\right](x;x') = \partial_\kappa\left\{i\kappa^2 a^{D-4}\sum_{I=1}^{6}U_I^{\mu\nu\kappa\lambda\alpha\beta\gamma\delta}\,i\left[{}_{\alpha\beta}\Delta_{\gamma\delta}\right](x;x)\,\partial_\lambda\delta^D(x-x')\right\}, \qquad (54)$$

where Table 1 lists the various $U_I^{\mu\nu\kappa\lambda\alpha\beta\gamma\delta}$. Once the vertex function terms have been inserted into the vacuum polarization the naive index contractions can be carried out term by term, the results of which are listed in Table 2.

I	$\partial_\kappa\left\{i\kappa^2 a^{D-4}U_I^{\mu\nu\kappa\lambda\alpha\beta\gamma\delta}\,i\Big[_{\alpha\beta}\Delta_{\gamma\delta}\Big](x;x)\,\partial_\lambda\delta^D(x-x')\right\}$
1	$\frac{1}{8}\eta^{\mu\nu}\partial_\kappa\left\{i\kappa^2 a^{D-4}i\Big[^\beta_{\ \beta}\Delta^\delta_{\ \delta}\Big]\partial^\kappa\delta^D\right\} - \frac{1}{8}\partial^\nu\left\{i\kappa^2 a^{D-4}i\Big[^\beta_{\ \beta}\Delta^\delta_{\ \delta}\Big]\partial^\mu\delta^D\right\}$
2	$-\frac{1}{4}\eta^{\mu\nu}\partial_\kappa\left\{i\kappa^2 a^{D-4}i\Big[^{\gamma\delta}\Delta_{\gamma\delta}\Big]\partial^\kappa\delta^D\right\} + \frac{1}{4}\partial^\nu\left\{i\kappa^2 a^{D-4}i\Big[^{\gamma\delta}\Delta_{\gamma\delta}\Big]\partial^\mu\delta^D\right\}$
3	$-\frac{1}{2}\eta^{\mu\nu}\partial_\kappa\left\{i\kappa^2 a^{D-4}i\Big[^\beta_{\ \beta}\Delta^{\kappa\lambda}\Big]\partial_\lambda\delta^D\right\} + \frac{1}{2}\partial^\nu\left\{i\kappa^2 a^{D-4}i\Big[^\beta_{\ \beta}\Delta^{\mu\lambda}\Big]\partial_\lambda\delta^D\right\}$ $-\frac{1}{2}\partial_\kappa\left\{i\kappa^2 a^{D-4}i\Big[^\beta_{\ \beta}\Delta^{\mu\nu}\Big]\partial^\kappa\delta^D\right\} + \frac{1}{2}\partial_\kappa\left\{i\kappa^2 a^{D-4}i\Big[^\beta_{\ \beta}\Delta^{\kappa\nu}\Big]\partial^\mu\delta^D\right\}$
4	$\partial_\kappa\left\{i\kappa^2 a^{D-4}i\Big[^{\mu\nu}\Delta^{\kappa\lambda}\Big]\partial_\lambda\delta^D\right\} - \partial_\kappa\left\{i\kappa^2 a^{D-4}i\Big[^{\lambda\mu}\Delta^{\kappa\nu}\Big]\partial_\lambda\delta^D\right\}$
5	$\eta^{\mu\nu}\partial_\kappa\left\{i\kappa^2 a^{D-4}i\Big[^{\lambda\delta}\Delta^\kappa_{\ \delta}\Big]\partial_\lambda\delta^D\right\} - \partial_\kappa\left\{i\kappa^2 a^{D-4}i\Big[^{\nu\delta}\Delta^\kappa_{\ \delta}\Big]\partial^\mu\delta^D\right\}$
6	$\partial_\kappa\left\{i\kappa^2 a^{D-4}i\Big[^{\mu\delta}\Delta^\nu_{\ \delta}\Big]\partial^\kappa\delta^D\right\} - \partial^\nu\left\{i\kappa^2 a^{D-4}i\Big[^{\mu\delta}\Delta^\lambda_{\ \delta}\Big]\partial_\lambda\delta^D\right\}$

Table 2: Terms of 4-point contribution after naive index contractions.

3.2 Substitution of Graviton Propagator

To complete the index contractions the full graviton propagator must be inserted. This will again create many more terms so it is useful to break the graviton propagator into three pieces and consider each part separately. Upon consideration of the graviton propagator (28) we see that if each of the three types of scalar propagators are set equal, to say $B(y)$, then the tensor components combine to give the conformal graviton propagator tensor component

$$\Big[_{\mu\nu}T^A_{\rho\sigma}\Big] + \Big[_{\mu\nu}T^B_{\rho\sigma}\Big] + \Big[_{\mu\nu}T^C_{\rho\sigma}\Big] = \Big[_{\mu\nu}T^{cf}_{\rho\sigma}\Big] = 2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \frac{2}{(D-2)}\eta_{\mu\nu}\eta_{\rho\sigma}\ . \tag{55}$$

By adding and subtracting $B(y)$ from each of the scalar propagators in (28) the graviton propagator can be rewritten as

$$i\Big[_{\mu\nu}\Delta_{\rho\sigma}\Big](x;x') \;=\; \Big[_{\mu\nu}T^{cf}_{\rho\sigma}\Big] \times B + \Big[_{\mu\nu}T^A_{\rho\sigma}\Big] \times (i\Delta_A - B) + \Big[_{\mu\nu}T^C_{\rho\sigma}\Big] \times (C - B)\ . \tag{56}$$

Writing the graviton propagator in this way cancels the conformal parts of the scalar propagators in the second two terms. This property is not useful for the 4-point contribution, but it will be helpful for renormalizing the 3-point contribution and we use it again here for consistency. We will refer to the three components of (56) separately as the conformal part, the A-type part and and C-type part as associated with the tensor factor of each piece.

Before making substitutions for the graviton propagator we need to know how each of the tensor components contract in the various ways appearing in Table 2. The relevant contractions are listed in Table 3, where each element represents the tensor factor in the top row contracted with the combination of flat space metrics listed in the left hand column.

	$\left[_{\mu\nu}T^{\rm cf}_{\rho\sigma}\right]$	$\left[_{\mu\nu}T^{A}_{\rho\sigma}\right]$	$\left[_{\mu\nu}T^{C}_{\rho\sigma}\right]$
$\eta^{\mu\nu}\eta^{\rho\sigma}$	$-\frac{4D}{(D-2)}$	$-\frac{4(D-1)}{(D-3)}$	$\frac{8}{(D-2)(D-3)}$
$\eta^{\mu\rho}\eta^{\nu\sigma}$	$\frac{D(D^2-D-4)}{(D-2)}$	$\frac{(D^3-4D^2+D+2)}{(D-3)}$	$2\frac{(D^2-5D+8)}{(D-2)(D-3)}$
$\eta^{\mu\nu}$	$-\frac{4}{(D-2)}\eta_{\rho\sigma}$	$-\frac{4}{(D-3)}\bar{\eta}_{\rho\sigma}$	$\frac{4}{(D-2)}\delta^0_\rho\delta^0_\sigma+\frac{4}{(D-2)(D-3)}\bar{\eta}_{\rho\sigma}$
$\eta^{\mu\rho}$	$\frac{(D^2-D-4)}{(D-2)}\eta_{\nu\sigma}$	$\frac{(D^2-3D-2)}{(D-3)}\bar{\eta}_{\nu\sigma}$	$-2\frac{(D-3)}{(D-2)}\delta^0_\nu\delta^0_\sigma+\frac{2}{(D-2)(D-3)}\bar{\eta}_{\nu\sigma}$

Table 3: Various graviton tensor factor contractions.

As stated above we dissect the graviton propagator substitution into three parts. The substitution and following index contractions for the conformal part of the graviton propagator are listed in Table 4, the terms resulting from the A-type part are listed in Table 5, and the terms from the C-type part in Table 6. Note that in these tables we suppress the $(x-x')$ factor on the delta functions for brevity. Upon further inspection of the conformal part of the graviton propagator in Table 4 we see that all six sets of terms have the same tensor structure. Combining all of the terms we find the simplified expression

$$
i\left[^{\mu}\Pi^{\nu}_{\rm 4pt}\right]_{\rm cf}(x;x') \;\; = \;\; -\frac{(D^3-9D^2+10D+16)}{4(D-2)}\left\{\eta^{\mu\nu}\partial_\kappa\left[i\kappa^2 a^{D-4}B(0)\partial^\kappa\delta^D(x-x')\right]\right.
$$
$$
\left.-\partial^\nu\left[i\kappa^2 a^{D-4}B(0)\partial^\mu\delta^D(x-x')\right]\right\} \; . \tag{57}
$$

Equation (57), and Tables 5 and 6 make up the 4-point contribution to the vacuum polarization. Next we will transform these results into the desired manifestly transverse form.

3.3   Finding the 4-Point Structure Functions

Recall that it is our goal to write the result for the vacuum polarization in the form of (40), where two transverse projection operators are acting on two scalar structure functions $F(x;x')$ and $G(x;x')$. Using the Tables of section 3.2 we can now derive these structure functions.

The easiest way we found to extract the structure functions was to exploit the known transversality of the vacuum polarization and isolate the structure functions via two contractions, resulting in two equations for the two structure functions. The first contraction is made with $\delta^0_\mu\delta^0_\nu$, applied to equation (40) this gives the identity

$$
i\left[^0\Pi^0\right](x;x') = \nabla^2 F(x;x') \; , \tag{58}
$$

which provides a simple equation for F. To find G the second contraction is taken with $\delta^i_\mu\delta^j_\nu(j\neq i)$, applied to (40) we find the second identity

$$
i\left[^i\Pi^j\right](x;x') = -\partial'^i\partial^j\left[F(x;x')+G(x;x')\right] \; . \tag{59}
$$

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I	$\partial_\kappa \left\{ i\kappa^2 a^{D-4} U_I^{\mu\nu\kappa\lambda\alpha\beta\gamma\delta} \, i\left[{}_{\alpha\beta}T^{\rm cf}_{\gamma\delta}\right] B(0)\, \partial_\lambda \delta^D(x-x') \right\}$
1	$-\frac{D}{2(D-2)} \left\{ \eta^{\mu\nu}\partial_\kappa \left[i\kappa^2 a^{D-4} B\partial^\kappa\delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4} B\partial^\mu\delta^D \right] \right\}$
2	$-\frac{D(D^2-D-4)}{4(D-2)} \left\{ \eta^{\mu\nu}\partial_\kappa \left[i\kappa^2 a^{D-4} B\partial^\kappa\delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4} B\partial^\mu\delta^D \right] \right\}$
3	$\frac{4}{(D-2)} \left\{ \eta^{\mu\nu}\partial_\kappa \left[i\kappa^2 a^{D-4} B\partial^\kappa\delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4} B\partial^\mu\delta^D \right] \right\}$
4	$-\frac{D}{(D-2)} \left\{ \eta^{\mu\nu}\partial_\kappa \left[i\kappa^2 a^{D-4} B\partial^\kappa\delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4} B\partial^\mu\delta^D \right] \right\}$
5	$\frac{(D^2-D-4)}{(D-2)} \left\{ \eta^{\mu\nu}\partial_\kappa \left[i\kappa^2 a^{D-4} B\partial^\kappa\delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4} B\partial^\mu\delta^D \right] \right\}$
6	$\frac{(D^2-D-4)}{(D-2)} \left\{ \eta^{\mu\nu}\partial_\kappa \left[i\kappa^2 a^{D-4} B\partial^\kappa\delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4} B\partial^\mu\delta^D \right] \right\}$

Table 4: 4-point contributions coming from the conformal part of the graviton propagator.

I	$\partial_\kappa \left\{ i\kappa^2 a^{D-4} U_I^{\mu\nu\kappa\lambda\alpha\beta\gamma\delta} \, i\left[{}_{\alpha\beta}T^{\rm A}_{\gamma\delta}\right] \left(i\Delta_A(x;x) - B(0) \right) \partial_\lambda \delta^D(x-x') \right\}$
1	$-\frac{(D-1)}{2(D-3)} \left\{ \eta^{\mu\nu}\partial_\kappa \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\partial^\kappa\delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\partial^\mu\delta^D \right] \right\}$
2	$-\frac{(D^3-4D^2+D+2)}{4(D-3)} \left\{ \eta^{\mu\nu}\partial_\kappa \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\partial^\kappa\delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\partial^\mu\delta^D \right] \right\}$
3	$\frac{2}{(D-3)} \left\{ \eta^{\mu\nu}\bar\partial_\kappa \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\bar\partial^\kappa\delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\bar\partial^\mu\delta^D \right] \right.$ $\left. +\bar\eta^{\mu\nu}\partial_\kappa \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\partial^\kappa\delta^D \right] - \bar\partial^\nu \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\partial^\mu\delta^D \right] \right\}$
4	$-\frac{(D-1)}{(D-3)} \left\{ \bar\eta^{\mu\nu}\bar\partial_\kappa \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\bar\partial^\kappa\delta^D \right] - \bar\partial^\nu \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\bar\partial^\mu\delta^D \right] \right\}$
5	$\frac{(D^2-3D-2)}{(D-3)} \left\{ \eta^{\mu\nu}\bar\partial_\kappa \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\bar\partial^\kappa\delta^D \right] - \bar\partial^\nu \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\partial^\mu\delta^D \right] \right\}$
6	$\frac{(D^2-3D-2)}{(D-3)} \left\{ \bar\eta^{\mu\nu}\partial_\kappa \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\partial^\kappa\delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4}(i\Delta_A - B)\bar\partial^\mu\delta^D \right] \right\}$

Table 5: 4-point contributions coming from the A-type part of the graviton propagator.

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I	$\partial_\kappa \left\{ i\kappa^2 a^{D-4} U_I^{\mu\nu\kappa\lambda\alpha\beta\gamma\delta} \, i \Big[_{\alpha\beta} T_{\gamma\delta}^C \Big] (C(0) - B(0)) \, \partial_\lambda \delta^D(x-x') \right\}$
1	$\frac{1}{(D-2)(D-3)} \left\{ \eta^{\mu\nu} \partial_\kappa \left[i\kappa^2 a^{D-4}(C-B)\partial^\kappa \delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4}(C-B)\partial^\mu \delta^D \right] \right\}$
2	$-\frac{(D^2-5D+8)}{2(D-2)(D-3)} \left\{ \eta^{\mu\nu} \partial_\kappa \left[i\kappa^2 a^{D-4}(C-B)\partial^\kappa \delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4}(C-B)\partial^\mu \delta^D \right] \right\}$
3	$\frac{2}{(D-2)} \left\{ \eta^{\mu 0} \partial^\nu \left[i\kappa^2 a^{D-4}(C-B)\partial^0 \delta^D \right] + \frac{1}{(D-3)} \partial^\nu \left[i\kappa^2 a^{D-4}(C-B)\bar{\partial}^\mu \delta^D \right] \right.$ $-\eta^{\mu\nu} \partial^0 \left[i\kappa^2 a^{D-4}(C-B)\partial^0 \delta^D \right] - \frac{1}{(D-3)} \eta^{\mu\nu} \bar{\partial}_\kappa \left[i\kappa^2 a^{D-4}(C-B)\bar{\partial}^\kappa \delta^D \right]$ $+\eta^{\nu 0} \partial^0 \left[i\kappa^2 a^{D-4}(C-B)\partial^\mu \delta^D \right] + \frac{1}{(D-3)} \bar{\partial}^\nu \left[i\kappa^2 a^{D-4}(C-B)\partial^\mu \delta^D \right]$ $\left. -\eta^{\mu 0}\eta^{\nu 0} \partial_\kappa \left[i\kappa^2 a^{D-4}(C-B)\partial^\kappa \delta^D \right] - \frac{1}{(D-3)} \bar{\eta}^{\mu\nu} \partial_\kappa \left[i\kappa^2 a^{D-4}(C-B)\partial^\kappa \delta^D \right] \right\}$
4	$\frac{2}{(D-2)(d-3)} \left\{ (D-3)\bar{\eta}^{\mu\nu} \partial^0 \left[i\kappa^2 a^{D-4}(C-B)\partial^0 \delta^D \right] + \bar{\eta}^{\mu\nu} \bar{\partial}_\kappa \left[i\kappa^2 a^{D-4}(C-B)\bar{\partial}^\kappa \delta^D \right] \right.$ $+(D-3)\eta^{\mu 0}\eta^{\nu 0} \bar{\partial}_\kappa \left[i\kappa^2 a^{D-4}(C-B)\bar{\partial}^\kappa \delta^D \right] - (D-3)\eta^{\nu 0} \partial^0 \left[i\kappa^2 a^{D-4}(C-B)\bar{\partial}^\mu \delta^D \right]$ $\left. -(D-3)\eta^{\mu 0} \bar{\partial}^\nu \left[i\kappa^2 a^{D-4}(C-B)\partial^0 \delta^D \right] - \bar{\partial}^\nu \left[i\kappa^2 a^{D-4}(C-B)\bar{\partial}^\mu \delta^D \right] \right\}$
5	$\frac{2}{(D-2)(D-3)} \left\{ -(D-3)^2 \eta^{\mu\nu} \partial^0 \left[i\kappa^2 a^{D-4}(C-B)\partial^0 \delta^D \right] - \bar{\partial}^\nu \left[i\kappa^2 a^{D-4}(C-B)\partial^\mu \delta^D \right] \right.$ $\left. +(D-3)^2 \eta^{\nu 0} \partial^0 \left[i\kappa^2 a^{D-4}(C-B)\partial^\mu \delta^D \right] + \eta^{\mu\nu} \bar{\partial}_\kappa \left[i\kappa^2 a^{D-4}(C-B)\bar{\partial}^\kappa \delta^D \right] \right\}$
6	$\frac{2}{(D-2)(D-3)} \left\{ \bar{\eta}^{\mu\nu} \partial_\kappa \left[i\kappa^2 a^{D-4}(C-B)\partial^\kappa \delta^D \right] - \partial^\nu \left[i\kappa^2 a^{D-4}(C-B)\bar{\partial}^\mu \delta^D \right] \right.$ $\left. -(D-3)^2 \eta^{\mu 0}\eta^{\nu 0} \partial_\kappa \left[i\kappa^2 a^{D-4}(C-B)\partial^\kappa \delta^D \right] + (D-3)^2 \eta^{\mu 0} \partial^\nu \left[i\kappa^2 a^{D-4}(C-B)\partial^0 \delta^D \right] \right\}$

Table 6: 4-point contributions coming from the C-type part of the graviton propagator.

Knowing F it is now trivial to find G.

It is true that there are other pairs of contractions that would work equally as well. For example, contracting with $\delta^0_\mu \delta^i_\nu$ and $\bar{\eta}_{\mu\nu}$ leads to the identities

$$i\left[{}^0\Pi^i\right](x; x') = -\partial'^0 \partial^i F(x; x') , \tag{60}$$

$$i\left[{}_{\bar{\mu}}\Pi^{\bar{\mu}}\right](x; x') = -(D-1)\partial^0 \partial'^0 F(x; x') - (D-2)\nabla^2 \left[F(x; x') + G(x; x')\right] . \tag{61}$$

These identities provide a useful check on the structure functions derived from identities (58) and (59). There are likely more contractions that would provide similar sets of equations, but these two sets were sufficient for our purposes.

It is quite a tedious task to show how applying these contractions plays out for the entire 4-point contribution, and the procedure is extremely repetitive. For the reader's sanity and our own we will work out one example and assume that the procedure for the rest of the 4-point contribution can be easily deduced. We will demonstrate how to find F and G from the conformal part of the graviton propagator as given in (57).

Contracting $\delta^0_\mu \delta^0_\nu$ with (57) we have

$$i\left[{}^0\Pi^0_{4\text{pt}}\right]_{\text{cf}}(x; x') = -\frac{(D^3 - 9D^2 + 10D + 16)}{4(D-2)}\left\{-\partial_\kappa \left[i\kappa^2 a^{D-4} B \partial^\kappa \delta^D(x - x')\right]\right.$$
$$\left. -\partial^0 \left[i\kappa^2 a^{D-4} B \partial^0 \delta^D(x - x')\right]\right\}$$
$$= -\frac{(D^3 - 9D^2 + 10D + 16)}{4(D-2)}\left\{(\partial \cdot \partial' + \partial^0 \partial'^0) \times \left[i\kappa^2 a^{D-4} B \delta^D(x - x')\right]\right\}$$
$$= \frac{(D^3 - 9D^2 + 10D + 16)}{4(D-2)}\nabla^2 \left[i\kappa^2 a^{D-4} B(0)\delta^D(x - x')\right] , \tag{62}$$

where in going from the first line to the second we used the delta function to make the change $\partial \to -\partial'$. It is then trivial to pull the inner derivative outside the curly brackets since all prefactors are only functions of x. Also the B-type propagator can be evaluated at $y = 0$ since it is being evaluated at coincidence ($x = x'$). Upon comparison with identity (58) we find the first structure function

$$F_{4,\text{cf}}(x; x') = \frac{(D^3 - 9D^2 + 10D + 16)}{4(D-2)}\left[i\kappa^2 a^{D-4} B(0)\delta^D(x - x')\right] . \tag{63}$$

To find the second structure function we contract $\delta^i_\mu \delta^j_\nu (j \neq i)$ with (57)

$$i\left[{}^i\Pi^j\right]_{\text{cf}}(x; x') = -\frac{(D^3 - 9D^2 + 10D + 16)}{4(D-2)}\left\{0 - \partial^j \left[i\kappa^2 a^{D-4} B \partial^i \delta^D(x - x')\right]\right\}$$
$$= -\frac{(D^3 - 9D^2 + 10D + 16)}{4(D-2)}\partial'^i \partial^j \left[i\kappa^2 a^{D-4} B \delta^D(x - x')\right] . \tag{64}$$

Invoking identity (59) and making the proper substitution for $F_{4,\text{cf}}$ it is easy to see

$$G_{4,\text{cf}}(x; x') = 0 . \tag{65}$$

This concludes the example case for finding F and G. The same procedure can be applied to all of the terms in Tables 5 and 6 to find the rest of the structure functions. Combining all F and G

contributions from the three parts of the graviton propagator produces the full result for the 4-point structure functions

$$F_4(x;x') \;=\; \kappa^2 a^{D-4}\left\{\frac{D(D-5)}{4}i\Delta_A(0) - \frac{(D-1)}{2}B(0) - \frac{(3D-10)}{2(D-2)}C(0)\right\}i\delta^D(x-x') \tag{66}$$

$$G_4(x;x') \;=\; -\kappa^2 a^{D-4}\left\{\frac{(D^2-4D+1)}{(D-3)}i\Delta_A(0) - (D-3)B(0) - 2\frac{(D-4)}{(D-3)}C(0)\right\}i\delta^D(x-x') \tag{67}$$

We can now make substitutions for the propagators. The B- and C-type propagators are a finite constant at coincidence in $D = 4$, but the A-type propagator is divergent. Therefore, it is useful to break the structure functions into their finite and divergent pieces

$$F_{4,\text{div}}(x;x') \;=\; -\frac{D(D-5)}{4}\frac{\kappa^2 a^{D-4}H^{D-2}\Gamma(D-1)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\pi\cot\left(\frac{\pi}{2}D\right)i\delta^D(x-x')\,, \tag{68}$$

$$F_{4,\text{finite}}(x;x') \;=\; \frac{\kappa^2 H^2}{4\pi^2}\left\{\frac{1}{4} - \ln(a)\right\}i\delta^4(x-x')\,, \tag{69}$$

$$G_{4,\text{div}}(x;x') \;=\; \frac{(D^2-4D+1)}{(D-3)}\frac{\kappa^2 a^{D-4}H^{D-2}\Gamma(D-1)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\pi\cot\left(\frac{\pi}{2}D\right)i\delta^D(x-x')\,, \tag{70}$$

$$G_{4,\text{finite}}(x;x') \;=\; -\frac{\kappa^2 H^2}{4\pi^2}\left\{\frac{1}{4} + \ln(a)\right\}i\delta^4(x-x')\,. \tag{71}$$

This concludes our derivation of the 4-point contributions to the structure functions, where we note that only the terms proportional to $\ln(a)$ in (68-71) cannot be absorbed into the counter terms. We will now derive the 3-point contribution.

4 The 3-Point Contribution

This section will cover the derivation of the contribution to the vacuum polarization attributed to the 3-point diagram in Fig. 1. The procedure for deriving the scalar structure functions from the diagram is very similar to the one used in section 3. First the naive index contractions are completed in pieces by dividing the 3-point vertex function appropriately. Then once the proper substitutions have been made for the graviton and photon propagators all remaining index contractions can be completed. Finally the 3-point contributions to F and G will be presented in several parts, broken up according to which pieces of the photon and graviton propagators the structure function originated from.

4.1 Naive contractions

Following the same organizing procedure as in the previous section, we complete the naive index contractions first by breaking the 3-point vertex function (18) into pieces, shown in Table 7, and rewriting the 3-point contribution as a sum of these terms

$$i\left[^{\mu}\Pi^{\nu}_{3\text{pt}}\right](x;x') \;=\; \partial_\kappa\partial'_\theta\Big\{i\kappa a^{D-4}\sum_{I=1}^{2}V_I^{\mu\rho\kappa\lambda\alpha\beta}i\left[_{\alpha\beta}\Delta_{\gamma\delta}\right](x;x')$$

$$\times i\kappa a'^{D-4}\sum_{J=1}^{2}V_J^{\nu\sigma\theta\phi\gamma\delta}\partial_\lambda\partial'_\phi i\left[_\rho\Delta_\sigma\right](x;x')\Big\}\,. \tag{72}$$

I	$V^{\mu\rho\kappa\lambda\alpha\beta}$
1	$\eta^{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu}$
2	$4\eta^{\alpha)[\mu}\eta^{\kappa][\rho}\eta^{\lambda](\beta}$

Table 7: Pieces of 3-point vertex function.

I	J	$-\partial_\kappa\partial'_\theta\left\{\kappa^2(aa')^{D-4}V_I^{\mu\rho\kappa\lambda\alpha\beta}i\left[_{\alpha\beta}\Delta_{\gamma\delta}\right](x;x')V_J^{\nu\sigma\theta\phi\gamma\delta}\partial_\lambda\partial'_\phi i\left[_\rho\Delta_\sigma\right](x;x')\right\}$
1	1	$-\kappa^2\partial_\kappa\partial'_\theta\left\{(aa')^{D-4}i\left[^\beta{}_\beta\Delta^\delta{}_\delta\right]\partial^{[\mu}\partial'^{[[\nu}i\left[^{\kappa]}\Delta^{\theta]]}\right]\right\}$
1	2	$\kappa^2\partial_\kappa\partial'_\theta\left\{(aa')^{D-4}i\left[^\beta{}_\beta\Delta^{\phi[\nu}\right]\partial^{[[\mu}\partial'_\phi i\left[^{\kappa]]}\Delta^{\theta]}\right]-(aa')^{D-4}i\left[^\beta{}_\beta\Delta^{\sigma[\nu}\right]\partial^{[[\mu}\partial'^{\theta]}i\left[^{\kappa]]}\Delta_\sigma\right]\right\}$
2	1	$\kappa^2\partial_\kappa\partial'_\theta\left\{(aa')^{D-4}i\left[^{\lambda[\mu}\Delta^\delta{}_\delta\right]\partial_\lambda\partial'^{[[\nu}i\left[^{\kappa]}\Delta^{\theta]]}\right]-(aa')^{D-4}i\left[^{\rho[\mu}\Delta^\delta{}_\delta\right]\partial^\kappa\partial'^{[\nu}i\left[_\rho\Delta^{\theta]}\right]\right\}$
2	2	$-\kappa^2\partial_\kappa\partial'_\theta\left\{(aa')^{D-4}i\left[^{\lambda[\mu}\Delta^{\phi[[\nu}\right]\partial_\lambda\partial'_\phi i\left[^{\kappa]}\Delta^{\theta]]}\right]-(aa')^{D-4}i\left[^{\rho[\mu}\Delta^{\phi[[\nu}\right]\partial^\kappa\partial'_\phi i\left[_\rho\Delta^{\theta]]}\right]\right.$ $\left.+(aa')^{D-4}i\left[^{\rho[\mu}\Delta^{\sigma[[\nu}\right]\partial^{\kappa]}\partial'^{\theta]]}i\left[_\rho\Delta_\sigma\right]-(aa')^{D-4}i\left[^{\lambda[\mu}\Delta^{\sigma[[\nu}\right]\partial_\lambda\partial'^{\theta]]}i\left[^{\kappa]}\Delta_\sigma\right]\right\}$

Table 8: 3-point results after naive index contractions.

(72) contains products of the parts of the two 3-point vertex functions; the result for completing the naive index contractions for all said products are shown in Table 8. To simplify the results in this table a short hand notation for antisymmetrization has been adopted, where both square brackets and double square brackets imply antisymmetrization. The index structure for these terms can be confusing and it should be noted that the antisymmetrization only applies to the immediate indices with in the brackets. Here is a worked out example

$$i\left[^{\rho[\mu}\Delta^{\sigma[[\nu}\right]\partial^{\kappa]}\partial'^{\theta]]} = \frac{1}{4}\left\{i\left[^{\rho\mu}\Delta^{\sigma\nu}\right]\partial^\kappa\partial'^\theta - i\left[^{\rho\kappa}\Delta^{\sigma\nu}\right]\partial^\mu\partial'^\theta + i\left[^{\rho\kappa}\Delta^{\sigma\theta}\right]\partial^\mu\partial'^\nu - i\left[^{\rho\mu}\Delta^{\sigma\theta}\right]\partial^\kappa\partial'^\nu\right\} . \quad (73)$$

To complete the index contractions, substitutions for the graviton and photon propagator must be made.

4.2   Graviton and Photon Propagator Substitutions

To make substitutions for the graviton and photon propagators and completing the index contractions as clear as possible we will be using the new form of the graviton propagator (56) and also break the photon propagator up in a similar manner. To modify the orginal photon propagator (37) we can again add and subtract $B(x;x')$ from each term. Rearranging we find

$$i\left[_\mu\Delta_\nu\right](x;x') = \eta_{\mu\nu}aa'B(y) + \delta_\mu^0\delta_\nu^0\left[B(y) - C(y)\right] . \quad (74)$$

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Since we will be considering different parts of the propagators individually for the rest of this section it is necessary to take a moment to explain our notation. We will refer to the first, second, and third term of (56) with the subscripts B, A, and C respectively. Likewise, for (74) we will refer to the first and second terms with the subscripts B and C respectively.

In the 3-point contribution it is always a product of the graviton and photon propagator that appear, so there are in general six combinations of terms that can arise BB, AB, CB, BC, AC, and CC, where the first letter stands for the part of the graviton propagator being considered and the second letter stands for the part of the photon propagator. However, it can be shown that the last two combinations do not need to be calculated since they are made of the product of two differences of scalar propagators. For these cases the conformal parts of the scalar propagators will cancel and what remains are only the infinite sums in each propagator, but these terms go like $\sim y^0$. Their degree of divergence is such that we can take $D = 4$ immediately, and in this case all of the infinite series vanish. It is true that the A-type scalar propagator has an extra term going like $\sim y^{2-D/2}$; however, in the $D = 4$ limit the infinite series multiplying this extra term will cause it to vanish too. So, in the end there are really only four combinations that need to be calculated.

One other notational comment is needed before the calculation can continue. All four possible combinations of the parts of the propagators will result in terms of the same form, namely

$$\# \partial_\alpha \partial'_\beta \left\{ (aa')^{D-4} i\Delta_x \partial_\gamma \partial'_\delta (aa' i\Delta_y) \right\} , \tag{75}$$

where $i\Delta_{x,y}$ is actually a difference of propagators for the cases $x = A, C$ or $y = C$. Since all of the terms will look almost identical, with only the propagators and indices changing, we can vastly simplify reporting our results by adopting a notation of only writing out the derivatives and their associated indices. These are the only parts needed because it is the indices that will make up the primary difference in each term, and the specific propagator combination can be denoted in the subscript of the vacuum polarization. In this notation (75) would take the simple form $\partial_\alpha \partial'_\beta (\partial_\gamma \partial'_\delta)$.

We are now ready to dive into the calculation. First we will perform the substitutions for the B part of the photon propagator with the B part of the graviton propagator. All of the terms for this portion will take the form

$$\# \partial_\alpha \partial'_\beta \left\{ (aa')^{D-4} B \partial_\gamma \partial'_\delta (aa' B) \right\} . \tag{76}$$

The full result for this set of propagator pieces is

$$i \left[{}^\mu \Pi^\nu_{\mathrm{3pt}} \right]_{\mathrm{B,B}} (x; x') = \frac{(D^2 - 4D + 2)}{(D-2)} \kappa^2 \Big\{ -\eta^{\mu\nu} \partial_\alpha \partial'_\beta (\partial^\alpha \partial'^\beta) + \partial^\nu \partial'_\beta (\partial^\mu \partial'^\beta) - \partial \cdot \partial' (\partial^\mu \partial'^\nu)$$
$$+ \partial_\alpha \partial'^\mu (\partial^\alpha \partial'^\nu) \Big\}$$
$$+ \kappa^2 \Big\{ -2\eta^{\mu\nu} \partial \cdot \partial' (\partial \cdot \partial') - \partial \cdot \partial' (\partial^\nu \partial'^\mu) + 2\partial^\nu \partial'^\mu (\partial \cdot \partial') + \partial_\alpha \partial'^\mu (\partial^\nu \partial'^\alpha)$$
$$- \eta^{\mu\nu} \partial_\alpha \partial'_\beta (\partial^\beta \partial'^\alpha) + \partial^\nu \partial'_\beta (\partial^\beta \partial'^\mu) \Big\} . \tag{77}$$

Next we compute the vacuum polarization from the B part of the graviton propagator and the C part of the photon propagator. These terms will all take the generic form

$$\# \partial_\alpha \partial'_\beta \left\{ (aa')^{D-4} B \partial_\gamma \partial'_\delta (aa'(B - C)) \right\} . \tag{78}$$

The full result is

$$i\left[{}^{\mu}\Pi^{\nu}_{\text{3pt}}\right]_{\text{B,C}}(x;x') = \frac{2}{(D-2)}\kappa^2\Big\{\eta^{\mu 0}\eta^{\nu 0}\partial_\alpha\partial'_\beta(\partial^\alpha\partial'^\beta) - \eta^{\nu 0}\partial^0\partial'_\beta(\partial^\mu\partial'^\beta) + \partial^0\partial'^0(\partial^\mu\partial'^\nu)$$

$$-\eta^{\mu 0}\partial_\alpha\partial'^0(\partial^\alpha\partial'^\nu)\Big\}$$

$$+\kappa^2\Big\{-\eta^{\mu\nu}\partial^0\partial'^0(\partial\cdot\partial') - \partial^0\partial'^0(\partial^\nu\partial'^\mu) + \eta^{\mu\nu}\partial_\alpha\partial'^0(\partial^\alpha\partial'^0) + \eta^{\mu\nu}\partial_\alpha\partial'_\beta(\partial^\alpha\partial'^\beta)$$

$$+\eta^{\mu\nu}\partial^0\partial'_\beta(\partial^0\partial'^\beta) + \eta^{\mu 0}\partial^\nu\partial'^0(\partial\cdot\partial') + \eta^{\mu 0}\partial_\alpha\partial'^0(\partial^\nu\partial'^\alpha) - \partial^\nu\partial'^0(\partial^\mu\partial'^0)$$

$$-\partial_\nu\partial'_\beta(\partial^\mu\partial'^\beta) - \eta^{\mu 0}\partial^\nu\partial'_\beta(\partial^0\partial'^\beta) - \eta^{\mu 0}\eta^{\nu 0}\partial\cdot\partial'(\partial\cdot\partial') - \eta^{\mu 0}\eta^{\nu 0}\partial_\alpha\partial'_\beta(\partial^\beta\partial'^\alpha)$$

$$+\partial\cdot\partial'(\partial^\mu\partial'^\nu) + \eta^{\nu 0}\partial^0\partial'^\mu(\partial\cdot\partial') + \eta^{\nu 0}\partial^0\partial'_\beta(\partial^\beta\partial'^\mu) - \eta^{\nu 0}\partial_\alpha\partial'^\mu(\partial^\alpha\partial'^0)$$

$$+\eta^{\nu 0}\partial\cdot\partial'(\partial^\mu\partial'^0) + \eta^{\mu 0}\partial\cdot\partial'(\partial^0\partial'^\nu) - \partial_\alpha\partial'_\nu(\partial^\alpha\partial'^\nu) \quad - \partial^0\partial'^\mu(\partial^0\partial'^\nu)\Big\}\ . \tag{79}$$

When we consider the A part of the graviton propagator and the B part of the photon propagator the terms all have the form

$$\#\partial_\alpha\partial'_\beta\left\{(aa')^{D-4}(i\Delta_A - B)\partial_\gamma\partial'_\delta(aa'B)\right\}\ , \tag{80}$$

and the full result is

$$i\left[{}^{\mu}\Pi^{\nu}_{\text{3pt}}\right]_{\text{A,B}}(x;x') = \frac{(D-1)(D-2)}{(D-3)}\kappa^2\Big\{\eta^{\mu\nu}\partial_\alpha\partial'_\beta(\partial^\alpha\partial'^\beta) + \bar{\partial}^\nu\partial'_\beta(\partial^\mu\partial'^\beta) - \bar{\partial}\cdot\bar{\partial}'(\partial^\mu\partial'^\nu)$$

$$+\partial_\alpha\bar{\partial}'^\mu(\partial^\alpha\partial'^\nu)\Big\}$$

$$+\kappa^2\Big\{-\eta^{\mu\nu}\partial\cdot\partial'(\bar{\partial}\cdot\bar{\partial}') - \partial\cdot\partial'(\bar{\partial}^\nu\bar{\partial}'^\mu) + \bar{\partial}^\nu\partial'\mu(\bar{\partial}\cdot\bar{\partial}') + \bar{\partial}_\alpha\partial'^\mu(\bar{\partial}^\nu\bar{\partial}'^\alpha)$$

$$-\eta^{\mu\nu}\bar{\partial}\cdot\bar{\partial}'(\partial\cdot\partial') - \eta^{\mu\nu}\bar{\partial}_\alpha\bar{\partial}'_\beta(\partial^\beta\partial'^\alpha) + \partial^\nu\bar{\partial}'^\mu(\bar{\partial}\cdot\bar{\partial}') + \partial^\nu\bar{\partial}'_\beta(\bar{\partial}^\beta\bar{\partial}'^\mu)\Big\}$$

$$+\frac{2}{(D-3)}\kappa^2\Big\{\partial_\alpha\partial'^\mu(\partial^\alpha\bar{\partial}'^\nu) - \eta^{\mu\nu}\partial_\alpha\bar{\partial}'_\beta(\partial^\alpha\bar{\partial}'^\beta) - \partial\cdot\partial'(\partial^\mu\bar{\partial}'^\nu) + \partial^\nu\bar{\partial}'_\beta(\partial^\mu\partial'^\beta)$$

$$+\partial^\nu\partial'_\beta(\bar{\partial}^\mu\partial'^\beta) - \eta^{\mu\nu}\bar{\partial}_\alpha\partial'_\beta(\bar{\partial}^\alpha\partial'^\beta) - \partial\cdot\partial'(\bar{\partial}^\mu\partial'^\nu) + \bar{\partial}_\alpha\partial'^\mu(\bar{\partial}^\alpha\partial'^\nu)$$

$$+\partial\cdot\partial'(\bar{\partial}^\mu\bar{\partial}'^\nu) - \bar{\partial}_\alpha\partial'^\mu(\bar{\partial}^\alpha\bar{\partial}'^\nu) + \eta^{\mu\nu}\bar{\partial}_\alpha\bar{\partial}'_\beta(\bar{\partial}^\alpha\bar{\partial}'^\beta) - \partial^\nu\bar{\partial}'_\beta(\bar{\partial}^\mu\bar{\partial}'^\beta)\Big\}$$

$$+\frac{(D-1)}{(D-3)}\kappa^2\Big\{\eta^{\mu\nu}\partial_\alpha\partial'_\beta(\partial^\alpha\partial'^\beta) - \partial^\nu\partial'_\beta(\partial^\mu\partial'^\beta) + \partial\cdot\partial'(\partial^\mu\partial'^\nu) - \partial_\alpha\partial'^\mu(\partial^\alpha\partial'^\nu)$$

$$+\bar{\eta}^{\mu\nu}\partial_\alpha\partial'_\beta(\partial^\alpha\bar{\partial}'^\beta) + \bar{\partial}\cdot\bar{\partial}'(\partial^\mu\bar{\partial}'^\nu) - \bar{\partial}^\nu\partial'_\beta(\partial^\mu\bar{\partial}'^\beta) - \partial_\alpha\bar{\partial}'^\mu(\partial^\alpha\bar{\partial}'^\nu)$$

$$-\bar{\partial}^\nu\partial'_\beta(\bar{\partial}^\mu\partial'^\beta) + \bar{\eta}^{\mu\nu}\bar{\partial}_\alpha\partial'_\beta(\bar{\partial}^\alpha\partial'^\beta - \bar{\partial}_\alpha\bar{\partial}'^\mu(\bar{\partial}^\alpha\partial'^\nu) + \bar{\partial}\cdot\bar{\partial}'(\bar{\partial}^\mu\partial'^\nu)\Big\}\ . \tag{81}$$

The last case is the C part of the graviton propagator and the B part of the photon propagator. These terms take the form

$$\#\partial_\alpha\partial'_\beta\left\{(aa')^{D-4}(C - B)\partial_\gamma\partial'_\delta(aa'B)\right\}\ , \tag{82}$$

and the full result is

$$i\Big[{}^{\mu}\Pi^{\nu}_{\mathrm{3pt}}\Big]_{\mathrm{C,B}}(x;x') = 2\frac{(D-1)}{(D-2)}\kappa^2\Big\{-\eta^{\mu 0}\partial_\alpha\partial'^0(\partial^\alpha\partial'^\nu)+\eta^{\nu 0}\eta^{\mu 0}\partial_\alpha\partial'_\beta(\partial^\alpha\partial'^\beta)+\partial^0\partial'^0(\partial^\mu\partial'^\nu)$$
$$-\eta^{\nu 0}\partial^0\partial'_\beta(\partial^\mu\partial'^\beta)\Big\}$$
$$+2\frac{(D-3)}{(D-2)}\kappa^2\Big\{-\eta^{\mu 0}\eta^{\nu 0}\partial\cdot\partial'(\partial^0\partial'^0)+\eta^{\nu 0}\partial^0\partial'^\mu(\partial^0\partial'^0)-\eta^{\mu\nu}\partial^0\partial'^0(\partial^0\partial'^0)+\eta^{\mu 0}\partial^\nu\partial'^0(\partial^0\partial'^0)\Big\}$$
$$+\frac{2}{(D-2)(D-3)}\kappa^2\Big\{-\eta^{\mu\nu}\partial_\alpha\partial'_\beta(\partial^\alpha\partial'^\beta)+\partial^\nu\partial'_\beta(\partial^\mu\partial'^\beta)-\partial\cdot\partial'(\partial^\mu\partial'^\nu)+\partial_\alpha\partial'^\mu(\partial^\alpha\partial'^\nu)$$
$$-\partial_\alpha\bar{\partial}'^\mu(\partial^\alpha\partial'^\nu)+\bar{\eta}^{\mu\nu}\partial_\alpha\partial'_\beta(\partial^\alpha\partial'^\beta)+\bar{\partial}\cdot\bar{\partial}'(\partial^\mu\partial'^\nu)-\bar{\partial}^\nu\partial'_\beta(\partial^\mu\partial'^\beta)$$
$$-\partial_\alpha\partial'^\mu(\partial^\alpha\bar{\partial}'^\nu)+\eta^{\mu\nu}\partial_\alpha\bar{\partial}'_\beta(\partial^\alpha\bar{\partial}'^\beta)+\partial\cdot\partial'(\partial^\mu\bar{\partial}'^\nu)-\partial^\nu\bar{\partial}'_\beta(\partial^\mu\bar{\partial}'^\beta)$$
$$-\partial^\nu\partial'_\beta(\bar{\partial}^\mu\partial'^\beta)+\eta^{\mu\nu}\bar{\partial}_\alpha\partial'_\beta(\bar{\partial}^\alpha\partial'^\beta)+\partial\cdot\partial'(\bar{\partial}^\mu\partial'^\nu)-\bar{\partial}_\alpha\partial'^\mu(\bar{\partial}^\alpha\partial'^\nu)$$
$$-\partial\cdot\partial'(\bar{\partial}^\mu\bar{\partial}'^\nu)+\partial_\alpha\bar{\partial}'^\mu(\partial^\alpha\bar{\partial}'^\nu)+\bar{\partial}^\nu\partial'_\beta(\bar{\partial}^\mu\partial'^\beta)+\bar{\partial}_\alpha\partial'^\mu(\bar{\partial}^\alpha\bar{\partial}'^\nu)$$
$$-\bar{\partial}\cdot\bar{\partial}'(\partial^\mu\bar{\partial}'^\nu)-\bar{\eta}^{\mu\nu}\bar{\partial}_\alpha\partial'_\beta(\bar{\partial}^\alpha\partial'^\beta)-\eta^{\mu\nu}\bar{\partial}_\alpha\bar{\partial}'_\beta(\bar{\partial}^\alpha\bar{\partial}'^\beta)+\bar{\partial}^\nu\partial'_\beta(\partial^\mu\bar{\partial}'^\beta)$$
$$+\bar{\partial}_\alpha\bar{\partial}'^\mu(\bar{\partial}^\alpha\partial'^\nu)+\partial^\nu\bar{\partial}'_\beta(\bar{\partial}^\mu\bar{\partial}'^\beta)-\bar{\eta}^{\mu\nu}\partial_\alpha\bar{\partial}'_\beta(\partial^\alpha\bar{\partial}'^\beta)-\bar{\partial}\cdot\bar{\partial}'(\bar{\partial}^\mu\partial'^\nu)\Big\}$$
$$+\frac{2}{(D-2)}\kappa^2\Big\{-\eta^{\nu 0}\partial_\alpha\partial'^\mu(\partial^\alpha\partial'^0)-\eta^{\mu 0}\partial^\nu\partial'_\beta(\partial^0\partial'^\beta)+\eta^{\mu\nu}\partial_\alpha\partial'^0(\partial^\alpha\partial'^0)+\eta^{\mu\nu}\partial^0\partial'_\beta(\partial^0\partial'^\beta)$$
$$+\eta^{\nu 0}\partial\cdot\partial'(\partial^\mu\partial'^0)+\eta^{\mu 0}\partial\cdot\partial'(\partial^0\partial'^\nu)-\partial^\nu\partial'^0(\partial^\mu\partial'^0)-\partial^0\partial'^\mu(\partial^0\partial'^\nu)$$
$$-\eta^{\mu 0}\partial\cdot\partial'(\partial^0\bar{\partial}'^\nu)-\eta^{\nu 0}\partial\cdot\partial'(\bar{\partial}^\mu\partial'^0)+\eta^{\mu 0}\partial_\alpha\partial'^0(\partial^\alpha\bar{\partial}'^\nu)+\eta^{\nu 0}\partial_\alpha\bar{\partial}'^\mu(\partial^\alpha\partial'^0)$$
$$+\eta^{\mu 0}\bar{\partial}^\nu\partial'_\beta(\partial^0\partial'^\beta)+\eta^{\nu 0}\partial^0\partial'_\beta(\bar{\partial}^\mu\partial'^\beta)+\partial^0\partial'^\mu(\partial^0\bar{\partial}'^\nu)+\eta^{\nu 0}\bar{\partial}_\alpha\partial'^\mu(\bar{\partial}^\alpha\partial'^0)$$
$$-\partial^0\partial'^0(\partial^\mu\bar{\partial}'^\nu)-\eta^{\nu 0}\bar{\partial}\cdot\bar{\partial}'(\partial^\mu\partial'^0)-\bar{\eta}^{\mu\nu}\partial^0\partial'_\beta(\partial^0\partial'^\beta)-\eta^{\nu 0}\eta^{\mu 0}\bar{\partial}_\alpha\partial'_\beta(\bar{\partial}^\alpha\partial'^\beta)$$
$$-\eta^{\mu\nu}\partial^0\bar{\partial}'_\beta(\partial^0\bar{\partial}'^\beta)-\eta^{\mu\nu}\bar{\partial}_\alpha\partial'^0(\bar{\partial}^\alpha\partial'^0)+\eta^{\nu 0}\partial^0\bar{\partial}'_\beta(\partial^\mu\bar{\partial}'^\beta)+\bar{\partial}^\nu\partial'^0(\partial^\mu\partial'^0)$$
$$+\partial^0\bar{\partial}'^\mu(\partial^0\partial'^\nu)+\eta^{\mu 0}\bar{\partial}_\alpha\partial'^0(\bar{\partial}^\alpha\partial'^\nu)+\eta^{\mu 0}\partial^\nu\bar{\partial}'_\beta(\partial^0\bar{\partial}'^\beta)+\partial^\nu\partial'^0(\bar{\partial}^\mu\partial'^0)$$
$$-\eta^{\mu 0}\eta^{\nu 0}\partial_\alpha\bar{\partial}'_\beta(\partial^\alpha\bar{\partial}'^\beta)-\bar{\eta}^{\mu\nu}\partial_\alpha\partial'^0(\partial^\alpha\partial'^0)-\eta^{\mu 0}\bar{\partial}\cdot\bar{\partial}'(\partial^0\partial'^\nu)$$
$$-\partial^0\partial'^0(\bar{\partial}^\mu\partial'^\nu)\Big\}\,.\tag{83}$$

4.3   Finding the 3-Point Structure Functions

Recalling the procedure outlined in section 3.3 we will now find the 3-point contributions to F and G in much the same way. The process will be a little more labor intensive since we no longer have a delta function on each term to assist in pulling out internal derivatives; there is also the added complication of having two internal derivatives instead of one. These changes will be accounted for as follows:

First we can still use the same identities (58) and (59) as our set of equations for finding F_3 and G_4. Next we notice that in the absence of a delta function we can no longer simply change $\partial \to -\partial'$, instead we have to carefully account for feed down terms that will arise from extracting derivatives. All of the 3-point terms take the form of (75), thus there are only four possible combinations of internal derivatives we will encounter, they are $\partial_i\partial'_j$, $\partial_0\partial'_i$, $\partial_i\partial'_0$, and $\partial_0\partial'_0$. Extracting these sets of derivatives will always result in the same feed down terms regardless of the propagators involved. A slight modification is needed for the A-type propagator, but in general extracting these derivatives

results in the following identities

$$
\begin{aligned}
(aa')^{D-4}i\Delta_x\partial_0\partial_i'(aa'i\Delta_y) &= \partial_0\partial_i'\left[(aa')^{D-4}I^2[i\Delta_x i\ddot{\Delta}_y]\right] + \partial_i'\Big\{H(aa')^{D-3}a\left(I[i\Delta_x i\dot{\Delta}_y]\right. \\
&\qquad\left. -(D-3)I^2[i\Delta_x i\ddot{\Delta}_y] + I^2[i\dot{\Delta}_x i\dot{\Delta}_y]\right)\Big\}\,,
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
(aa')^{D-4}i\Delta_x\partial_i\partial_0'(aa'i\Delta_y) &= \partial_i\partial_0'\left[(aa')^{D-4}I^2[i\Delta_x i\ddot{\Delta}_y]\right] + \partial_i\Big\{H(aa')^{D-3}a\left(I[i\Delta_x i\dot{\Delta}_y]\right. \\
&\qquad\left. -(D-3)I^2[i\Delta_x i\Delta_y''] + I^2[i\dot{\Delta}_x i\dot{\Delta}_y]\right)\Big\}\,,
\end{aligned}
\tag{85}
$$

$$
(aa')^{D-4}i\Delta_x\partial_i\partial_j'(aa'i\Delta_y) = \partial_i\partial_j'\left[(aa')^{D-4}I^2[i\Delta_x i\ddot{\Delta}_y]\right] - 2\eta_{ij}\left[H^2(aa')^{D-2}I[i\dot{\Delta}_x i\dot{\Delta}_y]\right]\,,
\tag{86}
$$

$$
\begin{aligned}
(aa')^{D-4}i\Delta_x\partial_0\partial_0'(aa'i\Delta_y) &= -2(D-3)H^2(aa')^{D-2}I[i\Delta_x i\dot{\Delta}_y] + \partial_0\partial_0'\left[(aa')^{D-3}I^2[i\Delta_x i\ddot{\Delta}_y]\right] \\
&\quad +(\partial_0 a' + \partial_0' a)\times\left\{H(aa')^{D-3}\left[I[i\Delta_x i\dot{\Delta}_y] - (D-3)I^2[i\Delta_x i\ddot{\Delta}_y]\right]\right\} \\
&\quad +\frac{1}{2}(aa')^{D-3}\nabla^2 I^3[i\dot{\Delta}_x i\dot{\Delta}_y] + H^2(aa')^{D-2}\Big\{i\Delta_x i\Delta_y + (2-y)I[i\dot{\Delta}_x i\dot{\Delta}_y] \\
&\quad -(D-1)I^2[i\dot{\Delta}_x i\dot{\Delta}_y] - (D-3)^2I^2[i\Delta_x i\ddot{\Delta}_y]\Big\}\,,
\end{aligned}
\tag{87}
$$

where I represents and an indefinite integral with respect to y, and a dot over the propagators represents a derivative with respect to y^1 . The A-type propagator is unique in that it is not just a function of y, but also contains a de Sitter breaking piece. It can be rewritten in the form

$$
i\Delta_A(x;x') = A(y) + ku\,,
\tag{88}
$$

where $k = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(D/2)}$. In this form we can see that identities (84-87) will miss the feed down terms that arise when one of the internal derivatives act on the second term in (88). To account for this the following terms need to be inserted in each of the above identities when $i\Delta_x = (i\Delta_A - B)$

$$
(aa')^{D-4}ku\partial_0\partial_i'(aa'B) \rightarrow -\partial_i'\left[Ha(aa')^{D-3}kB\right]
\tag{89}
$$

$$
(aa')^{D-4}ku\partial_i\partial_0'(aa'B) \rightarrow -\partial_i\left[Ha'(aa')^{D-3}kB\right]
\tag{90}
$$

$$
(aa')^{D-4}ku\partial_i\partial_j'(aa'B) \rightarrow 0
\tag{91}
$$

$$
\begin{aligned}
(aa')^{D-4}ku\partial_0\partial_0'(aa'B) \rightarrow &-(\partial_0 a' + \partial_0' a)\times\left[H(aa')^{D-3}kB\right] \\
&+2H^2(D-4)(aa')^{D-2}kB
\end{aligned}
\tag{92}
$$

We are now ready to apply the contractions in identities (58) and (59) and then use the substitutions (84-87), making use of (89-92) where appropriate, to find F_3 and G_3. Again, we will work through an example for the reader and then state the final result for all of the terms.

---

[1]One might worry about local delta function contributions from terms of the form $\partial_0\partial_0'B$, but actually these contributions are fully accounted for in delta functions arising from feed down terms of the form $\partial_0\partial_0'\left[(aa')^{D-3}I^2[i\Delta_x i\ddot{\Delta}_y]\right]$.

We consider all terms with the coefficient $\frac{(D^2-4D+2)}{(D-2)}\kappa^2$ in (77), since these form the smallest set of transverse terms. Performing the first contraction with $\delta_\mu^0\delta_\nu^0$ results in

$$
\begin{aligned}
i\left[{}^0\Pi_{3\text{pt,ex}}^0\right]_{B,B} &= \frac{(D^2-4D+2)}{(D-2)}\kappa^2\Big\{\partial_\alpha\partial_\beta'\left[(aa')^{D-4}B\partial^\alpha\partial'^\beta(aa'B)\right]\\
&\quad+\partial^0\partial_\beta'\left[(aa')^{D-4}B\partial^0\partial'^\beta(aa'B)\right]-\partial\cdot\partial'\left[(aa')^{D-4}B\partial^0\partial'^0(aa'B)\right]\\
&\quad+\partial_\alpha\partial'^0\left[(aa')^{D-4}B\partial^\alpha\partial'^0(aa'B)\right]\Big\}\\
&= \frac{(D^2-4D+2)}{(D-2)}\kappa^2\Big\{\partial_i\partial_j'\left[(aa')^{D-4}B\partial_i\partial_j'(aa'B)\right]\\
&\quad+\nabla^2\left[(aa')^{D-4}B\partial_0\partial_0'(aa'B)\right]\Big\}
\end{aligned}
\tag{93}
$$

where we have taken the 3+1 decomposition in going from the first line to the second. Now we can make the appropriate substitutions using identities (84-87) and compare with (58) to find

$$
\begin{aligned}
F_{3,\text{ex}}(x;x') &= \frac{(D^2-4D+2)}{(D-2)}\kappa^2\Big\{-2(D-3)H^2(aa')^{D-2}I[B\dot B]\\
&\quad+(\partial_0\partial_0'+\nabla^2)\left[(aa')^{D-3}I[B\ddot B]\right]\\
&\quad+(\partial_0a'+\partial_0'a)\times\left[H(aa')^{D-3}\Big(I[B\dot B]-(D-3)I^2[B\ddot B]\Big)\right]\\
&\quad+H^2(aa')^{D-2}\left[B^2+(4-y)I[\dot B^2]-(D-1)I^2[\dot B^2]-(D-3)^2I^2[B\ddot B]\right]\\
&\quad+\frac{1}{2}(aa')^{D-3}\nabla^2I^3[\dot B^2]\Big\}\,.
\end{aligned}
\tag{94}
$$

To find the companion $G_{3,\text{ex}}$ we contract $\delta_\mu^i\delta_\nu^j(i\neq j)$ with the same set of terms in (77)

$$
\begin{aligned}
i\left[{}^i\Pi_{3\text{pt,ex}}^j\right]_{B,B} &= \frac{(D^2-4D+2)}{(D-2)}\kappa^2\Big\{0+\partial^j\partial_\beta'\left[(aa')^{D-4}B\partial^i\partial'^\beta(aa'B)\right]\\
&\quad-\partial\cdot\partial'\left[(aa')^{D-4}B\partial^i\partial'^j(aa'B)\right]+\partial_\alpha\partial'^i\left[(aa')^{D-4}B\partial^\alpha\partial'^j(aa'B)\right]\Big\}\\
&= \frac{(D^2-4D+2)}{(D-2)}\kappa^2\Big\{-\partial_j\partial_0'\left[(aa')^{D-4}B\partial_i\partial_0'(aa'B)\right]+\partial_j\partial_k'\left[(aa')^{D-4}B\partial_i\partial_k'(aa'B)\right]\\
&\quad+\partial_0\partial_0'\left[(aa')^{D-4}B\partial_i\partial_j'(aa'B)\right]+\nabla^2\left[(aa')^{D-4}B\partial_i\partial_j'(aa'B)\right]\\
&\quad-\partial_0\partial_i'\left[(aa')^{D-4}B\partial_0\partial_j'(aa'B)\right]+\partial_k\partial_i'\left[(aa')^{D-4}B\partial_k\partial_j'(aa'B)\right]\,,
\end{aligned}
\tag{95}
$$

where again we have taken the 3+1 decomposition in going from the first line to the second. Applying identities (84-87), (95) can be rewritten in the form

$$
\begin{aligned}
i\left[{}^i\Pi_{3\text{pt,ex}}^j\right]_{B,B} &= -\frac{(D^2-4D+2)}{(D-2)}\kappa^2\partial_i'\partial_j\Big\{(\nabla^2-\partial_0\partial_0'-\partial_0'^2-\partial_0^2)\times\\
&\quad\left[(aa')^{D-3}I^2[B\ddot B]\right]\\
&\quad-\partial_0'\left[H(aa')^{D-3}a'\Big(I[B\dot B]-(D-3)I^2[B\ddot B]+I^2[\dot B]\Big)\right]\\
&\quad-\partial_0\left[H(aa')^{D-3}a\Big(I[B\dot B]-(D-3)I^2[B\ddot B]+I^2[\dot B]\Big)\right]\\
&\quad+4H^2(aa')^{D-2}I[\dot B^2]\Big\}\,.
\end{aligned}
\tag{96}
$$

Substituting $F_{3,\mathrm{ex}}$ into (59) we find

$$
\begin{aligned}
G_{3,\mathrm{ex}}(x;x') = {} & -\frac{(D^2-4D+2)}{(D-2)}\kappa^2\Big\{ H^2(aa')^{D-2}\Big(B^2-(D-3)^2I^2[B\ddot{B}] \\
& -(D-1)I^2[\dot{B}^2]-2(D-3)I[B\dot{B}]-yI[\dot{B}^2]\Big) \\
& +\frac{1}{2}(aa')^{D-3}\nabla^2 I^3[\dot{B}^2] \\
& +(\partial_0 a'+\partial_0' a)\times\Big[H(aa')^{D-3}\Big(I[B\dot{B}]-(D-3)I^2[B\ddot{B}]\Big)\Big] \\
& +\partial_0\Big[H(aa')^{D-3}a\Big(I[B\dot{B}]-(D-3)I^2[B\ddot{B}]+I^2[\dot{B}^2]\Big)\Big] \\
& +\partial_0'\Big[H(aa')^{D-3}a'\Big(I[B\dot{B}]-(D-3)I^2[B\ddot{B}]+I^2[\dot{B}^2]\Big)\Big] \\
& +(\partial_0+\partial_0')^2\Big[(aa')^{D-3}I^2[B\ddot{B}]\Big]\Big\}\ .
\end{aligned}
\tag{97}
$$

This concludes the 3-point example; it should be clear how to proceed with the rest of the 3-point contribution.

We now present the results for the scalar structure functions organized according to propagator combination. From (77) we find

$$
\begin{aligned}
F_{BB}(x;x') = {} & \frac{(D^2-D-4)}{(D-2)}\kappa^2\Big\{ H^2(aa')^{D-2}\Big[B^2-(D-3)^2I^2[B\ddot{B}]-(D-1)I^2[\dot{B}^2] \\
& -2(D-3)I[B\dot{B}]\Big]+\frac{1}{2}(aa')^{D-3}\nabla^2 I^3[\dot{B}^2]+\big(\nabla^2+\partial_0\partial_0'\big)\times\Big[(aa')^{D-3}I^2[B\ddot{B}]\Big] \\
& +H\,(\partial_0 a'+\partial_0' a)\times\Big[(aa')^{D-3}\Big(I[B\dot{B}]-(D-3)I^2[B\ddot{B}]\Big)\Big]\Big\} \\
& +\frac{\kappa^2 H^2}{(D-2)}(aa')^{D-2}\big[4D(2D-5)-(D^2-D-4)y\big]I[B^2]\ ,
\end{aligned}
\tag{98}
$$

$$
\begin{aligned}
G_{BB}(x;x') = {} & -\frac{\kappa^2}{(D-2)}\Big\{ D(D-3)H^2(aa')^{D-2}\Big[B^2-(D-3)^2I^2[B\ddot{B}]-(D-1)I^2[\dot{B}^2] \\
& -2(D-3)I[B\dot{B}]-yI[\dot{B}^2]\Big]+\frac{1}{2}D(D-3)(aa')^{D-3}\nabla^2 I^3[\dot{B}^2] \\
& -H\,(\partial_0 a'+\partial_0' a)\times\Big[(aa')^{D-3}\Big(I^2[\dot{B}^2]-(D^2-4D+2)(D-3)I^2[B\ddot{B}] \\
& +(D^2-4D+2)I[B\dot{B}]\Big)\Big] \\
& -(D^2-4D+2)H\partial_0\Big[a(aa')^{D-3}\Big(\frac{1}{(D-2)}I^2[\dot{B}^2]+(D-3)I^2[B\ddot{B}]-I[B\dot{B}]\Big)\Big] \\
& -(D^2-4D+2)H\partial_0'\Big[a'(aa')^{D-3}\Big(\frac{1}{(D-2)}I^2[\dot{B}^2]+(D-3)I^2[B\ddot{B}]-I[B\dot{B}]\Big)\Big] \\
& +(D^2-4D+2)(\partial_0+\partial_0')^2\Big[(aa')^{D-3}I^2[B\ddot{B}]\Big]\Big\}\ .
\end{aligned}
\tag{99}
$$

From (79) we find

$$F_{BC}(x;x') = -\frac{\kappa^2}{(D-2)}\Big\{2(D^2 - D - 4)H^2(aa')^{D-2}I[\dot{B}(\dot{B} - \dot{C})]$$
$$+ (3D - 8)(aa')^{D-3}\nabla^2 I^2[B(\ddot{B} - \ddot{C})]\Big\}, \tag{100}$$

$$G_{BC}(x;x') = 2\kappa^2\frac{(D-3)}{(D-2)}\Big\{DH^2(aa')^{D-2}I[\dot{B}(\dot{B} - \dot{C})]$$
$$+ (\nabla^2 + \partial_0\partial'_0) \times \Big[(aa')^{D-3}I^2[B(\ddot{B} - \ddot{C})]\Big]\Big\}. \tag{101}$$

From (81) we have

$$F_{AB}(x;x') = \kappa^2\Big\{H^2(D-1)(aa')^{D-2}\Big[B(i\Delta_A - B) - (D-3)^2 I^2[(i\Delta_A - B)\ddot{B}]$$
$$-(D-1)I^2[(i\dot{\Delta}_A - \dot{B})\dot{B}] - 2(D-3)I[(i\Delta_A - B)\dot{B}] + (4-y)I[(i\dot{\Delta}_A - \dot{B})\dot{B}]\Big]$$
$$+\frac{1}{2}(D-1)(aa')^{D-3}\nabla^2 I^3[(i\dot{\Delta}_A - \dot{B})\dot{B}]$$
$$+ (\nabla^2 + (D-1)\partial_0\partial'_0) \times \Big[(aa')^{D-3}I^2[(i\Delta_A - B)\ddot{B}]\Big]$$
$$+(D-1)H(\partial_0 a' + \partial'_0 a) \times \Big[(aa')^{D-3}\Big(I[(i\Delta_A - B)\dot{B}] - (D-3)I^2[(i\Delta_A - B)\ddot{B}]\Big)\Big]\Big\}$$
$$+(D-1)k\kappa^2\Big\{2(D-4)H^2(aa')^{D-2}B - H(\partial_0 a' + \partial'_0 a)\big[(aa')^{D-3}B\big]\Big\}, \tag{102}$$

$$G_{AB}(x;x') = -\kappa^2\Big\{H^2(aa')^{D-2}(D-1)\Big[B(i\Delta_A - B) - (D-3)^2 I^2[(i\Delta_A - B)\ddot{B}]$$
$$-(D-1)I^2[(i\dot{\Delta}_A - \dot{B})\dot{B}] - 2(D-3)I[(i\Delta_A - B)\dot{B}]$$
$$- \Big(4\frac{(D-4)}{(D-3)} + y\Big)I[(i\dot{\Delta}_A - \dot{B})\dot{B}]\Big]$$
$$+ \Big[-\frac{(D^2 - 4D + 1)}{(D-3)}\nabla^2 + (D-2)(\partial_0^2 + \partial_0'^2) + (D-1)\partial_0\partial'_0\Big] \times \Big[(aa')^{D-3}I^2[(i\Delta_A - B)\ddot{B}]\Big]$$
$$+\frac{1}{2}(D-1)(aa')^{D-3}\nabla^2 I^3[(i\dot{\Delta}_A - \dot{B})\dot{B}]$$
$$+ H(D-2)\partial'_0\Big[a'(aa')^{D-3}\Big(I^2[(i\dot{\Delta}_A - \dot{B})\dot{B}] + I[(i\Delta_A - B)\dot{B}]$$
$$-(D-3)I^2[(i\Delta_A - B)\ddot{B}]\Big)\Big]$$
$$+ H(D-2)\partial_0\Big[a(aa')^{D-3}\Big(I^2[(i\dot{\Delta}_A - \dot{B})\dot{B}] + I[(i\Delta_A - B)\dot{B}]$$
$$-(D-3)I^2[(i\Delta_A - B)\ddot{B}]\Big)\Big]$$
$$+ H(D-1)(\partial_0 a' + \partial'_0 a) \times \Big[(aa')^{D-3}\Big(I[(i\Delta_A - B)\dot{B}] - (D-3)I^2[(i\Delta_A - B)\ddot{B}]\Big)\Big]\Big\}$$
$$-k\kappa^2\Big\{2(D-4)(D-1)H^2(aa')^{D-2}B - (D-2)(\partial_0 a' + \partial'_0 a) \times \big[(aa')^{D-3}B\big]$$
$$-H(D-1)\partial'_0\big[a'(aa')^{D-3}B\big] - H(D-1)\partial_0\big[a(aa')^{D-3}B\big]\Big\}. \tag{103}$$

And lastly, from (83) we have

$$
\begin{aligned}
F_{CB}(x;x') = 2\frac{(D-3)}{(D-2)}\kappa^2\Big\{ &H^2(aa')^{D-2}\Big[B(C-B)-(D-3)^2I^2[(C-B)\ddot{B}] \\
&-(D-1)I^2[(\dot{C}-\dot{B})\dot{B}]-2(D-3)I[(C-B)\dot{B}] \\
&+(4-y)I[(\dot{C}-\dot{B})\dot{B}]\Big]+\frac{1}{2}(aa')^{D-3}\nabla^2I^3[(\dot{C}-\dot{B})\dot{B}] \\
&+H(\partial_0 a'+\partial_0' a)\times\Big[(aa')^{D-3}\Big(I[(C-B)\dot{B}]-(D-3)I^2[(C-B)\ddot{B}]\Big)\Big] \\
&+(\nabla^2+\partial_0\partial_0')\Big[(aa')^{D-3}I^2[(C-B)\ddot{B}]\Big]\Big\},
\end{aligned}
\tag{104}
$$

$$
\begin{aligned}
G_{CB}(x;x') = \frac{2\kappa^2}{(D-2)}\Big\{ &H^2(aa')^{D-2}\Big[-(D-3)B(C-B)+(D-3)^2I^2[(C-B)\ddot{B}] \\
&+(D-3)(D-1)I^2[(\dot{C}-\dot{B})\dot{B}]+2(D-3)^2I[(C-B)\dot{B}] \\
&-\Big(4\frac{(D-2)(D-4)}{(D-3)}-(D-3)y\Big)I[(\dot{C}-\dot{B})\dot{B}]\Big] \\
&-\Big[\frac{(D-4)(D-2)}{(D-3)}\nabla^2+2(D-3)\partial_0\partial_0'-\partial_0^2-\partial_0'^2\Big]\times\Big[(aa')^{D-3}I^2[(C-B)\ddot{B}]\Big] \\
&-\frac{1}{2}(D-3)(aa')^{D-3}\nabla^2I^3[(\dot{C}-\dot{B})\dot{B}] \\
&+H(D-3)(\partial_0 a'+\partial_0' a)\times\Big[(aa')^{D-3}\Big((D-3)I^2[(C-B)\ddot{B}]-I[(C-B)\dot{B}]\Big)\Big] \\
&-H\partial_0\Big[a(aa')^{D-3}\Big((D-3)I^2[(C-B)\ddot{B}]-I^2[(\dot{C}-\dot{B})\dot{B}] \\
&-I[(C-B)\dot{B}]\Big)\Big] \\
&-H\partial_0'\Big[a'(aa')^{D-3}\Big((D-3)I^2[(C-B)\ddot{B}]-I^2[(\dot{C}-\dot{B})\dot{B}] \\
&-I[(C-B)\dot{B}]\Big)\Big]\Big\}.
\end{aligned}
\tag{105}
$$

Because of the different combinations of propagators appearing in each set of F and G it was not useful or enlightening to combine them here. Instead, once they are renormalized in the next section they are combined easily.

5 Renormalization

This section is devoted to renormalizing the 3 and 4-point contributions to F and G. First, all of the 3-point contributions must be put in the same form so as to be easily combined. We can then localize the ultraviolet divergent pieces, and combine them with the 4-point divergences. By reading off the correct counterterm coefficents we remove all divergences, and are left with the fully renormalized structure functions of the vacuum polarization.

5.1 Converting to Functions of Δx

The expressions we derived in section 4 contain many indefinite integrals of products of derivatives of the three propagator functions $i\Delta_A(y)$, $B(y)$ and $C(y)$. Each of these products consists of a few

powers of $y(x; x')$ which are singular at coincidence ($x'^\mu = x^\mu$) and whose coefficients are nonzero for $D = 4$, plus an infinite series of less and less singular powers of y whose coefficients vanish for $D = 4$. Because the vacuum polarization is used inside the 4-dimensional integral of the quantum-corrected Maxwell equation (3), the only terms which require dimensional regularization are those which are at least as singular as $1/y^2$. Any less singular term can be evaluated for $D = 4$, at which point most of the tedious infinite series contributions vanish. Recalling as well that $y(x; x') = H^2 aa' \Delta x^2$, we can make the following simplifications:

For F_{BB} and G_{BB} we use the identities

$$B^2 \rightarrow \frac{\Gamma^2\left(\frac{D}{2} - 1\right)}{16\pi^D (aa')^{D-2}\Delta x^{2D-4}} \tag{106}$$

$$I[B\dot{B}] \rightarrow \frac{\Gamma^2\left(\frac{D}{2} - 1\right)}{32\pi^D (aa')^{D-2}\Delta x^{2D-4}} \tag{107}$$

$$I[\dot{B}^2] \rightarrow -\frac{(D-2)^2}{(D-1)}\frac{\Gamma^2\left(\frac{D}{2} - 1\right)}{64\pi^D (aa')^{D-2}\Delta x^{2D-4}} + \frac{(D-2)}{256\pi^D}\Gamma^2\left(\frac{D}{2} - 1\right)\frac{1}{(aa')^{D-2}\Delta x^{2D-4}} \tag{108}$$

$$I^2[\dot{B}^2] \rightarrow \frac{(D-2)}{(D-1)}\frac{\Gamma^2\left(\frac{D}{2} - 1\right)}{64\pi^D (aa')^{D-2}\Delta x^{2D-4}} \tag{109}$$

$$I^3[\dot{B}^2] \rightarrow \frac{H^2}{96\pi^4}\frac{1}{aa'\Delta x^2} \tag{110}$$

$$I^2[B\ddot{B}] \rightarrow \frac{D}{(D-1)}\frac{\Gamma^2\left(\frac{D}{2} - 1\right)}{64\pi^D (aa')^{D-2}\Delta x^{2D-4}} \tag{111}$$

where we note that for this set of identities only (110) could be put in the $D = 4$ limit. For F_{BC} and G_{BC} most of the terms will go to zero and there is only one identity needed

$$I[\dot{B}(\dot{B} - \dot{C})] \rightarrow -\frac{(D-4)}{128}\frac{\Gamma^2\left(\frac{D}{2} - 1\right)}{\pi^D (aa')^{D-2}\Delta x^{2D-4}} \,. \tag{112}$$

For F_{AB} and G_{AB} we find that we can take $D = 4$ in most of the terms

$$B(i\Delta_A - B) \rightarrow -\frac{H^2}{32\pi^4}\left[\frac{1}{2} + \ln\left(\frac{1}{4}H^2\Delta x^2\right)\right]\frac{1}{aa'\Delta x^2} \tag{113}$$

$$I[(i\Delta_A - B)\dot{B}] \rightarrow -\frac{H^2}{32\pi^4}\left[\frac{3}{2} + \ln\left(\frac{1}{4}H^2\Delta x^2\right)\right]\frac{1}{aa'\Delta x^2} \tag{114}$$

$$I[(i\dot{\Delta}_A - \dot{B})\dot{B}] \rightarrow -\frac{(D-2)}{2}\frac{\Gamma^2\left(\frac{D}{2} - 1\right)}{64\pi^D}\frac{1}{(aa')^{D-2}\Delta x^{2D-4}} \tag{115}$$

$$I^2[(i\dot{\Delta}_A - \dot{B})\dot{B}] \rightarrow \frac{H^2}{62\pi^4}\frac{1}{aa'\Delta x^2} \tag{116}$$

$$I^3[(i\dot{\Delta}_A - \dot{B})\dot{B}] \rightarrow \frac{H^2}{64\pi^4}\left[\ln\left(\frac{1}{4}H^2\Delta x^2\right) + u\right] \tag{117}$$

$$I^2[(i\Delta_A - B)\ddot{B}] \rightarrow -\frac{H^2}{32\pi^4}\left[2 + \ln\left(\frac{1}{4}H^2\Delta x^2\right)\right]\frac{1}{aa'\Delta x^2} \tag{118}$$

$$kB \rightarrow \frac{H^2}{32\pi^4}\frac{1}{aa'\Delta x^2} \tag{119}$$

Notice that (115) is the only identity that needs to be kept in D dimensions. Lastly for F_{CB} and G_{CB} there is again only one relevant identity since most of the terms will be zero

$$I[(\dot{C}-\dot{B})\dot{B}] \rightarrow \frac{(D-4)}{128}\frac{\Gamma^2\left(\frac{D}{2}-1\right)}{\pi^D(aa')^{D-2}\Delta x^{2D-4}} . \tag{120}$$

5.2 Finding the Finite and Divergent Parts of F and G

It is easiest to renormalize the structure functions term by term; thus we will demonstrate the procedure for one term and the reader can extrapolate from the example to derive the rest of the terms. First, it is useful to note the necessary identities for renormalization. The terms proportional to $1/\Delta x^2$ are already integrable and do not need to be renormalized. There are also terms proportional $1/\Delta x^{2D-2}$ and $1/\Delta x^{2D-4}$; these will need to be renormalized. We use dimensional regulation to partially integrate these terms until they are integrable

$$\frac{1}{\Delta x^{2D-2}} = \frac{\partial^4}{4(D-2)^2(D-3)(D-4)}\frac{1}{\Delta x^{2D-6}} , \tag{121}$$

$$\frac{1}{\Delta x^{2D-4}} = \frac{\partial^2}{2(D-3)(D-4)}\frac{1}{\Delta x^{2D-6}} , \tag{122}$$

but it is clear that these identities contain a divergence in the factors of $\frac{1}{(D-4)}$. We can localize the divergence by adding zero in the form

$$\partial^2\frac{1}{\Delta x^2} = \frac{i4\pi^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)}\delta^D(x-x') . \tag{123}$$

Then by adding (123) to the divergent parts of (121) and (122) we find

$$\frac{\partial^2}{(D-4)}\frac{1}{\Delta x^{2D-6}} = \frac{i4\pi^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)}\frac{\mu^{D-4}\delta^D(x-x')}{(D-4)} - \frac{\partial^2}{(D-4)}\left\{\frac{1}{\Delta x^{2D-6}}-\frac{\mu^{D-4}}{\Delta x^{D-2}}\right\}$$

$$= \frac{i4\pi^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)}\frac{\mu^{D-4}\delta^D(x-x')}{(D-4)} - \frac{\partial^2}{2}\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right]+\mathcal{O}(D-4) , \tag{124}$$

where the factor of μ is added for dimensional consistency. We may now begin our example:

We consider the eighth term in F_{BB}

$$F_{\text{ex}}(x;x') = \frac{(D^2-D-4)}{(D-2)}\kappa^2(\nabla^2+\partial_0\partial_0')\left[(aa')^{D-3}I^2[B\ddot{B}]\right] . \tag{125}$$

Applying identity (111) we find

$$F_{\text{ex}}(x;x') = \frac{(D^2-D-4)}{(D-2)}\kappa^2(\nabla^2+\partial_0\partial_0')\left[\frac{D}{64\pi^D(D-1)}\frac{\Gamma(\frac{D}{2}-1)}{aa'\Delta x^{2D-4}}\right] . \tag{126}$$

We can now use identities (122) and (124) to break this into its divergent and finite pieces. We find

$$F_{\text{ex,div}}(x;x') = \frac{D(D^2-D-4)}{8(D-1)(D-2)(D-3)}\frac{\kappa^2}{16\pi^D}\Gamma\left(\frac{D}{2}-1\right)\left[\frac{1}{a^2}\partial^2+2\frac{H}{a}\partial_0\right] \times$$

$$\left[\frac{i4\pi^{D/2}}{(D-4)}\mu^{D-4}\delta^D(x-x')\right] , \tag{127}$$

$$F_{\text{ex,finite}}(x;x') = -\frac{\kappa^2}{48\pi^4}(\nabla^2+\partial_0\partial_0')\times\left\{\frac{1}{aa'}\partial^2\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right]\right\} , \tag{128}$$

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where in deriving the divergent part we have used the fact that, in conjunction with the delta function, $\partial_0\partial'_0\frac{1}{aa'} \to 2\frac{H}{a}\partial_0 - \frac{1}{a^2}\partial_0^2$. Renormalizing all other terms will follow a very similar procedure.

Once all of the terms have been renormalized and combined we finally arrive at the full result for the structure functions coming from the 3-point diagram, for convenience we split the results into their finite and divergent pieces

$$F_{3,\text{div}}(x;x') = \left\{ \frac{(D^5 - 6D^4 + 6D^3 + 3D^2 + 32D - 56)}{8(D-1)(D-2)(D-3)(D-4)}H^2 + \frac{D}{8(D-4)(D-1)}\left[\frac{1}{a^2}\partial^2 + 2\frac{H}{a}\partial_0\right] \right\}$$
$$\times \frac{i\kappa^2}{4\pi^{D/2}}\Gamma\left(\frac{D}{2}-1\right)\mu^{D-4}\delta^D(x-x') ,$$
(129)

$$F_{3,\text{finite}}(x;x') = -\frac{\kappa^2}{192\pi^4 aa'}\partial^4\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right]$$
$$+\frac{\kappa^2 H^2}{16\pi^4}\left\{ \frac{5}{4}\partial^2\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right] - \frac{3}{2}\partial^2\left[\frac{2}{\Delta x^2} + \frac{\ln\left(\frac{1}{4}H^2\Delta x^2\right)}{\Delta x^2}\right] - \frac{\Delta\eta}{6}\partial_0\partial^2\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right] \right.$$
$$\left. +\nabla^2\left[\frac{7}{3\Delta x^2} + \frac{\ln\left(\frac{1}{4}H^2\Delta x^2\right)}{\Delta x^2}\right] \right\}$$
$$+\frac{3\kappa^2 H^4 aa'}{32\pi^2}\left\{ \frac{3}{\Delta x^2} + 2\frac{\ln\left(\frac{1}{4}H^2\Delta x^2\right)}{\Delta x^2} + \frac{\Delta\eta}{2}\partial_0\frac{1}{\Delta x^2} + \frac{1}{4}\nabla^2\ln(H^2\Delta x^2) \right\}$$
(130)

$$G_{3,\text{div}}(x;x') = \left\{ \frac{(D^5 - 5D^4 + 4D^3 + 11D^2 - 15D - 4)}{8(D-1)(D-2)(D-3)(D-4)} \right\} \times$$
$$\frac{i\kappa^2 H^2}{4\pi^{D/2}}\Gamma\left(\frac{D}{2}-1\right)\mu^{D-4}\delta^D(x-x') ,$$
(131)

$$G_{3,\text{finite}}(x;x') = -\frac{\kappa^2 H^2}{32\pi^4}\left\{ \frac{11}{2}\partial^2\frac{1}{\Delta x^2} + \frac{1}{2}\partial^2\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right] + \partial^2\left[\frac{\ln\left(\frac{1}{4}H^2\Delta x^2\right)}{\Delta x^2}\right] \right.$$
$$\left. -\frac{\Delta\eta}{12}\partial_0\partial^2\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right] + \frac{1}{3}\nabla^2\frac{1}{\Delta x^2} \right\}$$
$$+\frac{3\kappa^2 H^4 aa'}{64\pi^4}\left\{ -\frac{7}{\Delta x^2} - 4\frac{\ln\left(\frac{1}{4}H^2\Delta x^2\right)}{\Delta x^2} - \Delta\eta\partial_0\frac{1}{\Delta x^2} + \frac{1}{2}\nabla^2\ln(H^2\Delta x^2) \right\} .$$
(132)

In the flat space limit $H \to 0, a = a' = 1$ these structure functions recover the old result [12]. We are now ready to combine the 3-point and 4-point contributions to find the full vacuum polarization.

5.3   Our Full Result

We will now find the appropriate counter term coefficients such as to cancel all divergences, and be left with the full renormalized vacuum polarization. Recall that we can actually absorb all of the 4-point contribution, minus the terms proportional to $\ln(a)$, into the counterterms. Thus to find the full counterterm coefficients we simply have to add the divergent coefficients of the 3-point contribution

to the the 4-point contribution. From (68), (69), and (129) we see

$$\overline{C} = -\frac{\kappa^2}{16} \left\{ \frac{(D^5 - 9D^4 + 29D^3 - 55D^2 + 80D - 56)}{8(D-1)(D-2)(D-3)(D-4)} \times \frac{\Gamma\left(\frac{D}{2}-1\right)\mu^{D-4}}{\pi^{D/2}} \right.$$

$$\left. -D(D-5) \times \frac{H^{D-4}\Gamma(D-1)\pi}{(4\pi)^{D/2}\Gamma(\frac{D}{2})} \cot\left(\frac{\pi}{2}D\right) - \frac{1}{4\pi^2} \right\} , \tag{133}$$

from (129) we find

$$C_4 = \frac{D}{(D-4)(D-1)} \frac{\kappa^2\Gamma\left(\frac{D}{2}-1\right)}{128\pi^{D/2}}\mu^{D-4} , \tag{134}$$

and from (70), (71), and (131) we find

$$\Delta C = -\frac{\kappa^2}{16} \left\{ \frac{(D^5 - 6D^4 + 15D^3 - 25D^2 + 21D - 4)}{8(D-1)(D-2)(D-3)(D-4)} \times \frac{\Gamma\left(\frac{D}{2}-1\right)}{\pi^{D/2}}\mu^{D-4} \right.$$

$$\left. +\frac{(D^2 - 4D + 1)}{(D-3)} \times \frac{4H^{D-4}\Gamma(D-1)\pi}{(4\pi)^{D/2}\Gamma(\frac{D}{2})} \cot\left(\frac{\pi}{2}D\right) - \frac{1}{4\pi^2} \right\} . \tag{135}$$

Using these coefficients in the counterterms (52) and (53) we have sufficiently removed all divergences and arrive at the final renormalized form for the structure functions

$$F(x;x') = \frac{\kappa^2}{24\pi^2} \left\{ \frac{H}{a}\partial_0 - 5H^2\ln(a) - \ln(a)\left[\frac{1}{a}\partial^2 + 2\frac{H}{a}\partial_0\right] \right\} i\delta^4(x - x')$$

$$-\frac{\kappa^2}{192\pi^4 aa'}\partial^4\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right]$$

$$+\frac{\kappa^2 H^2}{16\pi^4} \left\{ \frac{1}{3}\left[\frac{17}{4}\partial^2 - \partial_0^2\right] \times \left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right] - \frac{3}{2}\partial^2\left[\frac{\ln\left(\frac{1}{4}H^2\Delta x^2\right)}{\Delta x^2}\right] + \frac{1}{24}\partial_0^2\partial^2\ln^2(\mu^2\Delta x^2) \right.$$

$$\left. +\nabla^2\left[\frac{7}{3\Delta x^2} + \frac{\ln\left(\frac{1}{4}H^2\Delta x^2\right)}{\Delta x^2}\right] - 12\pi^2 i\delta^4(x - x') \right\}$$

$$+\frac{3\kappa^2 H^4 aa'}{32\pi^2} \left\{ \frac{5}{2\Delta x^2} + \frac{1}{4}\partial^2\ln(H^2\Delta x^2) + 2\frac{\ln\left(\frac{1}{4}H^2\Delta x^2\right)}{\Delta x^2} \right\} , \tag{136}$$

$$G(x;x') = -\frac{5\kappa^2 H^2}{6\pi^2}\ln(a)i\delta^4(x - x') - \frac{\kappa^2 H^2}{32\pi^4} \left\{ 22\pi^2 i\delta^4(x - x') + \frac{1}{6}\left[\frac{7}{2}\partial^2 - \partial_0^2\right] \times \left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right] \right.$$

$$\left. +\partial^2\left[\frac{\ln\left(\frac{1}{4}H^2\Delta x^2\right)}{\Delta x^2}\right] + \frac{1}{48}\partial_0^2\partial^2\ln^2(\mu^2\Delta x^2) + \frac{1}{3}\nabla^2\frac{1}{\Delta x^2} \right\}$$

$$+\frac{3\kappa^2 H^4 aa'}{64\pi^4} \left\{ -\frac{6}{\Delta x^2} + \frac{1}{2}\left(\partial^2 + \partial_0^2\right)\ln(H^2\Delta x^2) - 4\frac{\ln\left(\frac{1}{4}H^2\Delta x^2\right)}{\Delta x^2} \right\} . \tag{137}$$

These structure functions comprise our main result.

6   Hartree Approximation

Solving the quantum-corrected Maxwell equation (3) is an exercise of comparable difficulty to the one we have just concluded, so it will appear in a separate publication. In the meantime, we can gain a qualitative understanding of what the result might show by applying the Hartree approximation [48]. This has been used previously to study the effect of charged inflationary scalars on photons [49] and (especially relevant to the current problem) the effect of inflationary gravitons on massless fermions [7]. In each of the previous cases the Hartree approximation gave the correct spacetime dependence of the one loop correction to the mode functions and the correct sign relative to the tree order result.

The Hartree approximation to (3) consists of replacing the Heisenberg operator field equation by its expectation value in free graviton vacuum and then expanding in terms of coincident graviton propagators,

$$0 = \partial_\mu \left\{ \sqrt{-g(x)}\, g^{\mu\rho}(x) g^{\nu\sigma}(x) F_{\rho\sigma}(x) \right\} , \tag{138}$$

$$\longrightarrow 0 = \partial_\mu \left\{ a^{D-4} \Big\langle \Omega_h \Big| \sqrt{-\widetilde{g}(x)}\, \widetilde{g}^{\mu\rho}(x)\widetilde{g}^{\nu\sigma}(x) \Big| \Omega_h \Big\rangle F_{\rho\sigma}(x) \right\} , \tag{139}$$

$$= \partial_\mu \left\{ a^{D-4} F^{\mu\nu} \right\} + \frac{\kappa^2}{2} \partial_\mu \left\{ U^{\mu\nu\rho\sigma\alpha\beta\gamma\delta} a^{D-4} i \Big[{}_{\alpha\beta}\Delta_{\gamma\delta} \Big](x;x) F_{\rho\sigma}(x) \right\} + \dots \tag{140}$$

Here $U^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}$ denotes the tensor factor of the 2-graviton-2-photon vertex, given in expression (19). Most of the coincident graviton propagator is a divergent constant; the secular effects for which we are searching derive from only the logarithm part of the A-type propagator (25). At this point we can also take $D = 4$,[2] so our Hartree approximation to the effective field equation (3) is,

$$0 = \partial_\mu F^{\mu\nu} + \frac{\kappa^2 H^2}{8\pi^2} \partial_\mu \left\{ U^{\mu\nu\rho\sigma\alpha\beta\gamma\delta} \Big[{}_{\alpha\beta} T^A_{\gamma\delta} \Big] \times \ln(a) F_{\rho\sigma} \right\} + O(\kappa^4) . \tag{141}$$

Recall that the A-type tensor factor was defined in expression (32).

Substituting (19) and (32) in (141) gives a simple result,

$$0 = \partial_\mu F^{\mu\nu} + \frac{\kappa^2 H^2}{4\pi^2} \partial_\mu \left\{ \ln(a) \Big[-3F^{\mu\nu} + 4F^{\overline{\mu}\nu} + 4F^{\mu\overline{\nu}} - 3F^{\overline{\mu}\overline{\nu}} \Big] \right\} + O(\kappa^4) . \tag{142}$$

Here a barred index on any tensor indicates that its 0-component has been suppressed, for example, $V^{\overline{\mu}} \equiv \overline{\eta}^{\mu\nu} V_\nu = V^\mu - \delta^\mu_0 V^0$. We can distinguish in expression (142) between the cases of $\nu = 0$ and $\nu = i$. The constraint equation is effectively multiplied by a secular factor,

$$0 = \partial_j F^{j0} + \frac{\kappa^2 H^2}{4\pi^2} \partial_j \left\{ \ln(a) F^{j0} \right\} + O(\kappa^4) , \tag{143}$$

$$= \left\{ 1 + \frac{\kappa^2 H^2}{4\pi^2} \ln(a) \right\} \partial_j F^{j0} + O(\kappa^4) . \tag{144}$$

---

[2] One might worry about factors of $\ln(a)$ which could arise when a divergent constant multiplies a^{D-4}. However, we can see from expressions (52) and (53) that the very same factor of a^{D-4} multiplies the counterterms which absorb divergences from the 4-point diagram. So there can be no secular contributions from this source and we may as well drop the divergent constants and take $D = 4$.

This has no effect on dynamical photons although it would lead to a secular screening of a point charge of the sort reported by Kitamoto and Kitazawa [51] provided there is no compensating secular factor on the charge density. The equation of relevance for dynamical photons is $\nu = i$,

$$\partial_\mu F^{\mu i} + \frac{\kappa^2 H^2}{4\pi^2} \left\{ \partial_0 \left[\ln(a) F^{0i} \right] + \partial_j \left[2 \ln(a) F^{ji} \right] \right\} + O(\kappa^4) . \tag{145}$$

The one loop correction to the effective field equation of course fixes only the one loop corrections to the field strength. We therefore expand in powers of κ^2,

$$F^{\mu\nu} = F_{(0)}^{\mu\nu} + \kappa^2 F_{(1)}^{\mu\nu} + \kappa^4 F_{(2)}^{\mu\nu} + \dots \tag{146}$$

Equations (144) and (145) imply the following relations for the one loop field strengths,

$$\partial_j \kappa^2 F_{(1)}^{j0} = 0 , \tag{147}$$

$$\partial_\mu \kappa^2 F_{(1)}^{\mu i} = -\frac{\kappa^2 H^2}{4\pi^2} \left\{ \partial_0 \left[\ln(a) F_{(0)}^{0i} \right] + \partial_j \left[2 \ln(a) F_{(0)}^{ji} \right] \right\} . \tag{148}$$

With the $U(1)$ Bianchi identity, the leading secular behavior is,

$$\kappa^2 F_{(1)}^{0i} \longrightarrow -\frac{\kappa^2 H^2}{4\pi^2} \ln(a) \times F_{(0)}^{0i} , \tag{149}$$

$$\kappa^2 F_{(1)}^{ij} \longrightarrow -\frac{\kappa^2 H^2}{4\pi^2} \frac{\ln(a)}{Ha} \times \left[\partial^i F_{(0)}^{0j} - \partial^j F_{(0)}^{0i} \right] . \tag{150}$$

We see that the one loop correction to the electric field strength of a photon tends to cancel its classical value whereas the one loop correction to the magnetic field strength dies off.

7 Discussion

We have used dimensional regularization to compute the one loop quantum gravitational contribution to the vacuum polarization on de Sitter background. We first calculated the 4-point contribution in section 3, and then derived the much more cumbersome 3-point contribution in section 4. Each result was expressed in the form (40) as the sum of two transverse projection operators acting on structure functions. In sub-section 2.5 the relevant BPHZ counterterms (47) were also reduced to this form, resulting in expressions (52-53). Renormalization was implemented in section 5 to give our final results (136) and (137) for the structure functions $F(x; x')$ and $G(x; x')$.

Our ultimate goal is to study how inflationary gravitons affect electrodynamics using the quantum-corrected Maxwell equation (3). Specializing to de Sitter in conformal coordinates, substituting our form (40) for representing the vacuum polarization, and partially integrating the primed derivatives, allows us to express the quantum-corrected Maxwell equations as,

$$\partial_\nu F^{\nu\mu}(x) + \partial_\nu \int d^4x' \left\{ iF(x; x') F^{\nu\mu}(x') + iG(x; x') F^{\bar\nu\bar\mu}(x') \right\} = J^\mu . \tag{151}$$

(Recall that a barred index is purely spatial.) Equation (151) can be employed the same way one uses the classical Maxwell equation to study dynamical photons ($J^\mu = 0$ solutions) and the electric

and magnetic fields induced by standard sources. We have already done this for the one vacuum polarization from gravitons on flat space background [12]. Closely related studies have also been made of the effects that inflationary scalars have on dynamical photons [3] and on electrodynamic forces [4].

The actual implementation of this program requires solving integro-differential equations in the context of the Schwinger-Keldysh formalism [50]. That is an project comparable to the one we have just completed, so it will be deferred to a separate work. However, a simple estimate of what it might give was derived in section 6 by making the Hartree approximation [48] to localize the effective field equation. We find that the one loop electric field strength (149) of dynamical photons experiences a secular growth which tends to cancel its free field value, whereas the one loop correction to the magnetic field strength (150) dies away compared to its free field value. It is interesting to note that the magnetic response to inflationary scalars also seems to be subdominant to the electric repsonse [4].

Working out what the full equation (151) gives for dynamical photons is important to check the observation in [9] that the spin-spin interaction between gravitons and fermions seems to explain why inflationary gravitons cause the fermion mode function to grow [7] whereas they have no secular effect on the mode function of a massless, minimally coupled scalar [8]. Another important exercise is to work out the effect of inflationary gravitons on the electric field of a point charge. This is the natural way to check the surprising claim of Kitamoto and Kitazawa that infrared gravitons screen sub-horizon interactions during inflation [51].

Before closing, we should comment on the gauge issue. The vacuum polarization requires fixing both the $U(1)$ and diffeomorphism symmetries, and the manner in which this is accomplished can affect the result. Our previous study of gravitons on flat background revealed no dependence upon the choice of electromagnetic gauge, but a huge variation with the gravitational gauge [12]. We believe there is not likely to be any gauge dependence in the leading secular infrared effects one finds from de Sitter gravitons because the spin two part of the graviton propagator has the same infrared logarithm term in any gauge [30, 28]. Note that it is perfectly possible for a 1PI function such as the vacuum polarization to change with the gauge, while a particular feature of its dependence on space and time is the same in all gauges [52]. That is precisely what happens with the pole terms of 1PI functions in flat space quantum field theory, and we suspect that the same applies for the leading secular dependence on de Sitter.

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